Affine plane curves with one place at infinity

Masakazu Suzuki *

Introduction

Let C be an irreducible algebraic curve in the complex affine plane $\boldsymbol{C}^2$. We shall say that C has *one place at infinity*, if the normalization of C is analytically isomorphic to a compact Riemann surface punctured by one point.

There are several works concerning the classification problem of the affine plane curves with one place at infinity to find the canonical models of these curves under the polynomial transoformations of the coordinates of $\boldsymbol{C}^2$.

In the case when C is non-singular and simply connected, Abhyankar-Moh[1] and Suzuki[11] proved independently that C can be transformed into a line by a polynomial coordinate transformation of $\boldsymbol{C}^2$. Namely, in this case, we can take a line as a canonical model.

In case C is singular and simply connected, Zeidenberg-Lin[12] proved that C has the canonical model of type $y^q = x^p$, where p and q are coprime integers ≥ 2.

Assuming that C is non-singular, the cases when the genus g of C is 2, 3 and 4 were studied by Neumann[8], A'Campo-Oka[3] from the topological view point and by Miyanishi[4] from the algebrico-geometric view point. Miyanishi[5] classified the dual graphs of the curves which appears by the minimal resolution of the singularity at infinity. Recently, Nakazawa-Oka[7] gave the classification of all the canonical models for the cases $g \leq 7$, and in its appendix, Nakazawa gave the classification for $g \leq 16$ without proof.

In the present paper, we shall first explain how to get the canonical compactification (M, E) of $\boldsymbol{C}^2$ corresponding to the minimal resolution of the singularity of the curve C at infinity (after taking the coordinate system of $\boldsymbol{C}^2$ which minimize the degree of the defining equation of C), and then we shall study the dual graph $\Gamma(E)$ of the boundary curve E.

In this way, we shall first get a new simple proof to the above mentioned Abhyankar-Moh-Suzuki theorem. Next, we shall make it clear the relationship between the dual graph $\Gamma(E)$ and the δ-sequence of Abhyankar-Moh theory. We shall get a new algebrico-geometric proof to the beautifull so-called semigroup theorem of Abhyankar-Moh [2] and its inverse theorem due to Sathaye-

*Graduate School of Mathematics, Kyushu University 36, Hakozaki, Fukuoka, 812, Japan

Stenerson[9]. Our new proof gives us also an algorithm to compute the weights of the dual graph $\Gamma(E)$ by computer[1] .

To end this introduction, I would like to express here my hearty thanks to Mr.Takashi Oishi and Mr.Koichi Koide for their help at the begining of this research to the experimental calculation of various algebraic invariants of the dual graphs by computer.

1 Preliminaries

1.1 Primitive Polynomials

Let $f(x,y)$ be a polynomial function on the complex affine plane $\boldsymbol{C}^2$ with coordinate system x and y. $f(x,y)$ will be called *primitive* if the algebraic curve defined by $f(x,y)=\alpha$ in $\boldsymbol{C}^2$ is irreducible for all complex numbers α except for a finite number of α's. The following proposition is well known (see for example the Appendix of Furushima[4]).

Proposition 1 *For any polynomial $f(x,y)$, there exists a primitive polynomial $F(x,y)$ and a polynomial $\varphi(z)$ of one variable z such that $f(x,y)=\varphi(F(x,y))$.*

From this proposition, we can get the following corollary.

Corollary. *Irreducible polynomials are always primitive.*

Infact, let us write $f(x,y)$ in the form

$$f(x,y)=\varphi(F(x,y))$$

by a primitive polynomial $F(x,y)$ and a polynomial $\varphi(z)$ of one variable z. Since the curve $C : f(x,y)=0$ is irreducible and f takes the zero of order 1 on C, $\varphi(z)$ vanishes only on $z=0$ and takes the zero of order 1 on $z=0$. Therefore $\varphi(z)$ is a polynomial of degree 1. This implies that f is primitive.

1.2 Dual graph

We shall assume from now that M is a non-singular projective algebraic surface over the complex number field and E an algebraic curve on M. We shall assume further that each irreducible component of E is non-singular and intersect each other at only one point at most. In such case, we shall say that E is *of normal crossing type.*

For a curve E of normal crossing type, we represent each irreducible component of E by a *vertex* and join the vertices if and only if the corresponding

[1] We implemented a program to get the list of the δ-sequences and the corresponding dual graphs $\Gamma(E)$ of the curves with one place at infinity with any given genus. We thus confirmed the classification table given by Nakazawa in [7].

irreducible components intersect each other. We associate to each vertices an integer, called *weight*, equal to the self-intersection number of the corresponding irreducible component on M. The weighted graph thus obtained will be called the *dual graph* of E and noted by $\Gamma(E)$. In case the values of the weights are not in question, the weights may be ommitted in the picture of the dual graphs bellow.

Lemma 1 *Let $E_1, \cdots, E_r, E_{r+1}$ be the irreducible components of E and assume that the dual graph $\Gamma(E)$ is of the following linear type:*

$$\Gamma(E)\ :\quad \begin{array}{cccccccc} -n_1 & & -n_2 & & -n_r & & & \\ \circ & \text{———} & \circ & \text{- - - - - -} & \circ & \text{———} & \circ & \\ E_1 & & E_2 & & E_r & & E_{r+1} & \end{array} \qquad (n_i \geq 2).$$

Assume further that there exists a holomorphic function f on a neighborhood U of $E_1 \cup E_2 \cup \cdots \cup E_r$ such that the zero divizor (f) of f on U is written in the following form.

$$(f) = \sum_{i=1}^{r} m_i E_i + m_{r+1} E_{r+1} \cap U.$$

Then,

(1) $m_2, \cdots, m_{r+1}$ are all multiple of m_1.

(2) Set $p_i = m_i/m_1$, $(1 \leq i \leq r+1)$, then (p_{r+1}, p_r) are coprime each other and the following continuous fraction expansion holds.

$$\frac{p_{r+1}}{p_r} = n_r - \cfrac{1}{n_{r-1} - \cdots - \cfrac{1}{n_2 - \cfrac{1}{n_1}}}$$

Proof. Since $(f) \cdot E_i = 0$, we have $m_{i+1} = n_i m_i - m_{i-1}$ for $i = 1, \cdots, r$, where $m_0 = 0$. The two assertions of the lemma are the immediate consequences of these equations.

1.3 Intersection matrix.

Let $E_1, E_2, \cdots, E_R$ be the irreducible components of E and consider the intersection matrix

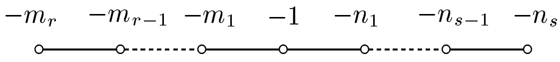
$$I_E = ((E_i \cdot E_j))_{i,j=1,\cdots R}.$$

Set $\Delta_E = \det(-I_E)$. The following two lemmas can also be obtained easily by a direct computation.

Lemma 2 *The determinant Δ_E is invariant under the blowing up of the points on E. Namely, if $\tau : M_1 \to M$ is a blowing up of a point P on E, we have then $\Delta_{\tau^{-1}(E)} = \Delta_E$.*

Lemma 3 *Assume that the dual graph* $\Gamma(E)$ *is of the following type:*

$\Gamma(E)$:



$(m_i \geq 2,\ n_j \geq 2)$

We have then

$$\Delta_E = pq - aq - bp.$$

where $p,\ a,\ q,\ b$ *are the natural numbers defined by the continuous fractions*

$$\frac{p}{a} = m_1 - \lfloor m_2 - \lfloor m_3 - \cdots - \lfloor m_r$$

$$\frac{q}{b} = n_1 - \lfloor n_2 - \lfloor n_3 - \cdots - \lfloor n_s$$

satisfying

$$(p, a) = 1,\ (q, b) = 1,\ 0 < a < p,\ 0 < b < q.$$

1.4 Compactifications of the affine plane

Assume now that $M - E$ is biregular to $\boldsymbol{C}^2$. In such a case, we shall call the pair $(M,\ E)$ an *algebraic compactification* of $\boldsymbol{C}^2$ and E the *boundary curve*. Let $E_1, E_2, \cdots, E_R$ be the irreducible components of E.

By C.P.Ramanujam[9] and J.A.Morrow[6], $(M,\ E)$ can be transformed into the pair $(\boldsymbol{P}^2,\ L)$ of the complex projective plane $\boldsymbol{P}^2$ and a line L on it, by a finitely many times of blowing ups and downs along the boundary curve. Therefore, by Lemma 2, $\Delta_E = -(L^2) = -1$, namely

$$\det I_E = \pm 1.$$

This implies that, for any R number of integers $k_1,\ k_2,\ \cdots,\ k_R$, there exists uniquely determined R integers $m_1,\ m_2,\ \cdots,\ m_R$ such that

$$\sum_{i=1}^{R} m_i(E_i \cdot E_j) = k_j, \quad (j = 1, 2, \cdots, R).$$

Thus, we have

Lemma 4 *Let* $(M,\ E)$ *be an algebraic compactification of* $\boldsymbol{C}^2$ *such that the boundary curve* E *is of normal crossing type and* $E_1,\ E_2,\ \cdots,\ E_R$ *be the irreducible components of* E*. Then, for any* R *number of integers* $k_1,\ k_2,\ \cdots,\ k_R$*, there exists a divisor* $D = \sum_{i=1}^{R} m_i E_i$ *with support on* E*, uniquely determined, such that*

$$(D \cdot E_j) = k_j, \quad (j = 1,\ 2,\ \cdots,\ R).$$

In particular, if $k_1, k_2, \cdots, k_R$ have a common divisor d, then all the coefficients $m_1, m_2, \cdots, m_R$ are multiple of d.

2 Resolution of the Singularity at Infinity

2.1 Canonical Coordinates

Let C be an irreducible affine algebraic curve with one place at infinity defined by a polynomial equation $f(x,y)=0$ in the complex affine plane $\boldsymbol{C}^2$ with the coordnate system $x,\ y$. Assume that the degree of $f(x,y)$ is m with respect to x and n with respect to y. Then, the usual argument about the Newton boundary shows that $f(x,y)$ is of the following form

$$f(x,y)=(ax^p+by^q)^d+\sum_{qi+pj<pqd} c_{ij}x^iy^j,$$

where $a\neq 0,\ b\neq 0,\ d=\gcd(m,n),\ p=m/d,\ q=n/d$.

In case $q=1$ (namely, $n=d,\ m=pn$), one can reduce the degree of $f(x,y)$ with respect x by a coordinate transformation of the following form

$$x_1=x, \quad y_1=y+cx^p$$

called *de Jonquière type.* Therefore, by a finitely many times of de Jonquière type coordinate transformations and the exchange of the coordinates x and y, one can reduce the polynomial $f(x,y)$ to one of the following two cases:

(A) $m = 1,\ n = 0$ (In this case, C is a line);

(B) $m=pd,\ n=qd,\ (p,q)=1,\ 1<q<p$.

Definition. *We shall call the coordinate system $x,\ y$ satisfying* (A) *or* (B) *the* canonical coordinate system *for C. An affine plane curve with one place at infinity having the canonical coordinate system of type* (A) *will be said* linealizable.

Assumption. *We shall assume, from now on to the end of this paper, that $f(x,y)$ is of type* (B) (non-linealizable type).

Now, let us compactify the plane $\boldsymbol{C}^2$ to get the projective plane $\boldsymbol{P}^2$ with the inhomogeneous coordinates $x,\ y$, and blow up the point at infinity of the curve C.

At the beginning, the closure $\overline{C}$ of C passes through the intersection point of the x-axis and the ∞-line A in $\boldsymbol{P}^2$, by the assumption $q<p$. Let us denote by E_0 the (-1)-curve appeared by the blowing up. The function $\dfrac{y^q}{x^p}$ has the

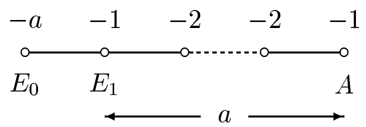
pole of order q on E_0 and the zero of order $p-q$ on A. Then, if $p-q>q$ (resp. $p-q<q$), $\overline{C}$ is tangent to E_0 (resp. A). Here, we denote the proper image of A by the same character A.

Let a be the positive integer defined by

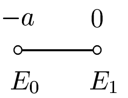
$$aq < p < (a+1)q.$$

In case $a=1$ $(p-q<q)$, we set $E_1 = A$.

In case $a>1$, after further $a-1$ times of the blowing ups of the point at infinity of the curve C, we get the compactification of $\boldsymbol{C}^2$ with the boundary curve having the following dual graph:



Here, the function $\dfrac{y^q}{x^p}$ has the pole of order q on E_0 and the zero of order $p-aq$ on E_1. Therefore, the closure of the curve C passes through the intersection point of E_0 and E_1, and is tangent to E_1. Now, blowing down from A the (-1)-curve on the right hand side of the dual graph $a-1$ times successively, we get the following dual graph:



$(a \;\geq 1)$.

Let $(M_1, E_0 \cup E_1)$ be the compactification of $\boldsymbol{C}^2$ thus obtained. The function f has the poles of order n on E_0 and of order m on E_1. The indetermination point of f on M_1 is uniquely the intersection point of E_0 and E_1.

Remark. *Note that, if one continue to blow up the the point at infinity of C at least twice, then the self-intersection numbers of the proper transforms of E_0 and E_1 become both ≤ -2, since $\overline{C}$ is tangent to E_1 on M_1.*

2.2 Successive Blowing ups

Now, from M_1, let us blow up the indetermination points of f successively, until the indetermination points of f disappear. Let M_f be the surface obtained by this resolution of the indetermination point of f. We shall continue to denote by E_0, E_1 the proper images of E_0, E_1 in M_f respectively, and let E_i $(2 \leq i \leq R)$ be the proper image in M_f of the (-1)-curve appeared by the $(i-1)$-th blowing up. Each E_i is a non-singular rational curve and the total curve $E_f = E_0 \cup E_1 \cup \cdots \cup E_R$ is of normal crossing type. The last curve E_R is of the first kind and f is non-constant on E_R. Note further that the union

$$E_1{}' = E_0 \cup E_2 \cup \cdots \cup E_R$$

is an exceptional set, since E_0 was exceptional in M_1.

Let us denote by $\overline{C}$ the closure of C in M_f. Let Z (resp. P) be the union of the components of E_f on which $f = 0$ (resp. $f = \infty$). Since f has no indetermination point on M_f, the zero $\overline{C} \cup Z$ and the pole P of f do not intersect each other. Let S be the union of the other components of E_f. f is non-constant on each irreducible component of S. We have $E_R \subset S$.

Suppose that P is *not connected*. Then, the connected component of P which does not contain E_1 must be exceptional, which is absurd. Hence, P is connected. P coincide with the connected component of $\overline{E_f - S}$ which contains E_0 and E_1, since f has no indetermination point on M_f.

In the same way, since Z is contained in the exceptional set ${E_1}'$, each connected component of Z must have an intersection point with $\overline{C}$. Since, on the other hand, C is of one place at infinity, $\overline{C}$ has only one intersection point with E_f. Therefore, Z is connected. Let Q be the intersection point of $\overline{C}$ and E_f.

(a) If $Q \notin Z$, then $Z = \emptyset$. In this case, S is irreducible, since each irreducible component of S must intersect $\overline{C}$.

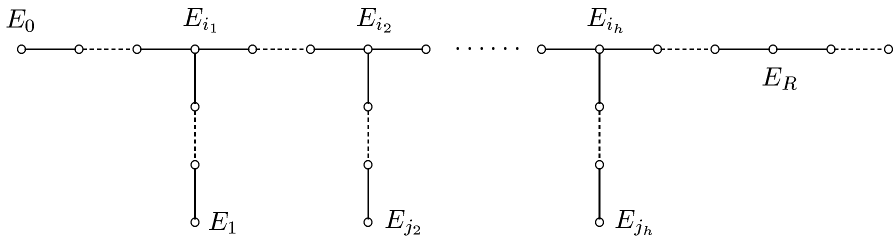
(b) If $Q \in Z$, then the zero points of f on S are necessarily on Z. Each irreducible component of S intersects Z and P which are both connected. Since the dual graph $\Gamma(E_f)$ of E_f is a tree, this implies that S is irreducible.

Thus, in either case S is irreducible, namely $S = E_R$.

Proposition 2 (i) *E_R is the unique irreducible component of E_f on which f is non-constant.* (ii) *P is the connected component of $\overline{E_f - E_R}$ which contains E_0 and E_1.*

Corollary 1 *At each step of the process to get M_f from M_1, the indetermination point of the function f to be blown up is unique, and is situated on the (-1)-curve appeared by the preceding blowing up.*

By this corollary, we see that the the dual graph $\Gamma(E_f)$ of E_f has the the following form:



Here, $1 = j_1 < i_1 < j_2 < i_2 < \cdots < j_h < i_h < R$. Taking into account the remark at the end of the previous subsection, we have

Corollary 2 $E_i^2 \leq -2$ *for* $i = 0, ..., R-1$ *and* $E_R^2 = -1$.

Since P is the connected component of $\overline{E_f - E_R}$ which contains E_0 and E_1, P is the union of the irreducible components of E_f situated in the left side to the vertex E_R in the above dual graph $\Gamma(E_f)$. Let

$$P_f = \sum_{i=0}^{R} \nu_i E_i$$

be the pole divisor of f on M_f. We have $\nu_i > 0$ for $E_i \subset P$ and $\nu_i = 0$ for the other components E_i. Let k be the degree of f on E_R. Then, we have

$$(*) \qquad (P_f \cdot E_i) = \begin{cases} 0 & (i \neq R) \\ k & (i = R). \end{cases}$$

Assume that $k > 1$. Then, by Lemma 4, $\nu_0, \cdots, \nu_R$ are all multiple of k. Since P is simply connected, there exists a simply connected neighborhood $U = U(P)$ of P such that $U \cap \overline{C} = \emptyset$, and there exists a meromorphic function F on U such that

$$f = F^k.$$

This implies that, for any complex number α with sufficiently large $|\alpha|$, the curve defined by $f = \alpha$ is composed of k irreducible components. This is a contradiction, since f is primitive by Proposition 1. Thus we get $k = 1$.

Suppose now that $\overline{E_f - P - E_R} \neq \emptyset$. Let $B_1, B_2, \cdots, B_s$ be the ireducible components of $B = \overline{E_f - P - E_R}$ in the order from the nearest to E_R (from left to right in the above dual graph $\Gamma(E_f)$). Let $(B_i^2) = -\beta_i$ $(1 \leq i \leq s)$ and set

$$n_1 = 1,\ n_2 = \beta_1,\ n_3 = n_2\beta_2 - n_1,\ \cdots,\ n_{s+1} = n_s\beta_s - n_{s-1}.$$

Since $\beta_i \geq 2$ for all i, we have $n_{s+1} \geq 2$. Consider the divisor $N = \sum_{i=1}^{s} n_i B_i$. We have then

$$(N \cdot E_i) = \begin{cases} 0 & (E_i \neq E_R, B_s) \\ 1 & (E_i = E_R) \\ -n_{s+1} & (E_i = B_s). \end{cases}$$

Since $k = 1$ in $(*)$, we get

$$((P_f - N) \cdot E_i) = \begin{cases} 0 & (E_i \neq B_s) \\ n_{s+1} & (E_i = B_s). \end{cases}$$

Therefore, by Lemma 4, all the coefficients of the divisor $P_f - N$ must be multiple of $n_{s+1} (\geq 2)$. This is a contradiction, since the coefficient of $P_f - N$ for B_1 is -1. Therefore, the components $B_1, \cdots, B_s$ do not exist. Thus, we have

Theorem 1 (i) *The degree of f on E_R is equal to* 1.

(ii) *The vertex corresponding to E_R is situated on an edge of $\Gamma(E_f)$.*

(iii) *Any irreducible component of E_f except for E_R is a pole of f.*

For each $\alpha \in \boldsymbol{C}$, denote by C_α the curve defined by $f = \alpha$ in $\boldsymbol{C}^2$. We have then

Corollary 1 *For any $\alpha \in \boldsymbol{C}$, the closure $\overline{C_\alpha}$ of C_α in M_f intersects E_R transversely at only one point (so is smooth at the intersection point), and intersects no other irreducible component E_i $(i \neq R)$. In particular, C_α is also irreducible and has one place at infinity.*

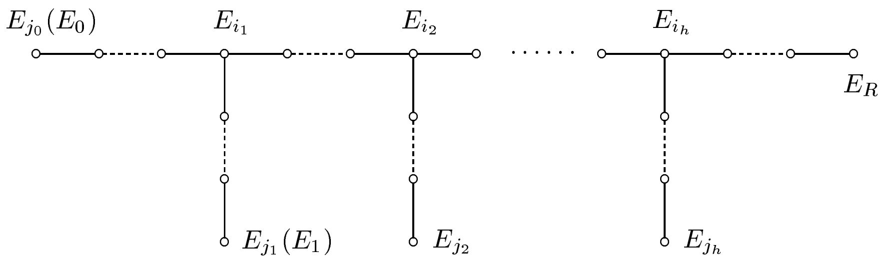
Corollary 2 *If $C = C_0$ is non-singular and of genus g, then any C_α except for a finite number of α's is of genus g.*

In particular, if $C = C_0$ is non-singular and simply connected, then all the curves $\overline{C_\alpha}$ are isomorphic to $\boldsymbol{P}^1$. The mapping $f : M_f \to \boldsymbol{P}^1$ defines a ruled surface structure on M_f, and $P = E_0 \cup E_1 \cup \cdots E_{R-1}$ is its singular fiber. Therefore, P must contain at least one (-1)-curve in its irreducible components. This is a contradiction, since E_0, E_1, $\cdots$, E_{R-1} contain no (-1)-curve by Corollary 2 of Proposition 2. Therefore, the case (B) which we assumed at the beginning of this section does not occur for simply connected non-singular C. Thus, we get

Corollary 3([1],[11]) *If an affine plane curve C is non-singular and simply connected, then there exists a polynomial coordinate transformation of $\boldsymbol{C}^2$ which maps C to a line in $\boldsymbol{C}^2$.*

2.3 Minimal Resolution of the Singularity at Infinity

By Theorem 1 (ii), the dual graph $\Gamma(E_f)$ is of the following form:



Let $i_1 < i_2 < \cdots < i_h$ be the indices of the irreducible components of E_f corresponding to the branching vertices of the graph $\Gamma(E_f)$, and $j_0 = 0 < j_1 = 1 < j_2 < \cdots < j_h < j_{h+1} = R$ be the indices corresponding to the edges of $\Gamma(E_f)$ as above.

Reversing the process of the construction of (M_f, E_f), one can blow down successively E_R, E_{R-1}, $\cdots$, E_2 in this order to get the smooth surface M_1. Therefore, we have $(E_i^2) = -2$, for $i_h \leq i \leq R-1$.

Definition. *Let $M = M_C$ be the surface obtained by the blowing down of E_R, E_{R-1}, $\cdots$, E_{i_h+1} from M_f. We shall denote the images of E_0, E_1, $\cdots$, E_{i_h} in M_C newly by the same notations E_0, E_1, $\cdots$, E_{i_h}, and set*

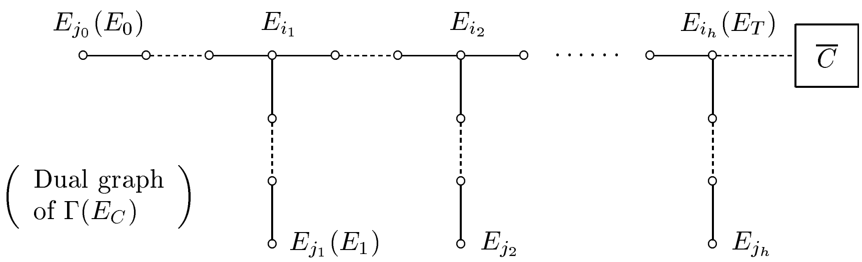
$$E = E_C = E_0 \cup E_1 \cup \cdots \cup E_{i_h} \text{ in } M_C.$$

We shall call the pair $(M, E) = (M_C, E_C)$ the compactification of $\boldsymbol{C}^2$ obtained by the minimal resolution *of the singularity of C at infinity. Accordingly, the graph $\Gamma(E_C)$ will be called the* dual graph of the minimal resolution *of the singularity of C at infinity.*

Setting

$$L_k = \bigcup_{i_{k-1} < i \leq i_k} E_i$$

for each $1 \leq k \leq h$, we shall call L_k (resp. $\Gamma(L_k)$) the *k-th branch* of E_C (resp. of $\Gamma(E_C)$), where $i_0 = -1$. We shall denote i_h by T.



3 $(p,\ q)$-sequence and δ-sequence

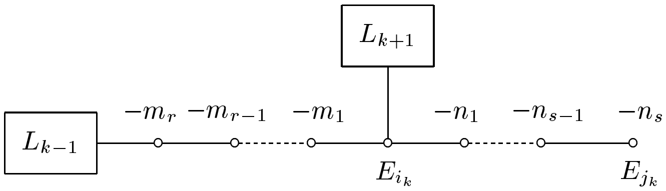
In the followings, we shall denote (M_C, E_C) by (M, E) and the closure of C in M by $\overline{C}$.

Definition. *We shall denote by δ_k the order of the pole of f on E_{j_k} for $0 \leq k \leq h$ and call $\{\delta_0, \delta_1, \cdots, \delta_h\}$ the* δ-sequence *of C (or of f).*

The purpose of this section is to describe the relationship between the δ-sequence and the weights of $\Gamma(E)$. Note first that we have

$$\delta_0 = n, \quad \delta_1 = m,$$

since we assumed that $f(x, y)$ is of degree m with respect to x and n with respect to y. For each k $(1 \leq k \leq h)$, set the weights of $\Gamma(L_k)$ as follows:



(k-th branch L_k),

$$(m_i \geq 2,\ n_j \geq 2)$$

where $L_0 =$ the closure of the y-axis and $L_{h+1} = \overline{C}$. Define the positive integers $p_k,\ a_k,\ q_k,\ b_k$ such that

$$(p_k, a_k) = 1,\ (q_k, b_k) = 1, \quad 0 < a_k < p_k,\ 0 < b_k < q_k,$$

by the continuous fractions as follows:

$$p_k/a_k \ = \ m_1 \ - \ \rfloor\!\!\!\lceil \ m_2 \ - \ \rfloor\!\!\!\lceil \ m_3 \ - \ \cdots \ - \ \rfloor\!\!\!\lceil \ m_r$$

$$q_k/b_k \ = \ n_1 \ - \ \rfloor\!\!\!\lceil \ n_2 \ - \ \rfloor\!\!\!\lceil \ n_3 \ - \ \cdots \ - \ \rfloor\!\!\!\lceil \ n_s.$$

Remark. In case there is no vertex between the two branching vertices corresponding to $E_{i_{k-1}}$ and E_{i_k} $(k > 1,\ r = 0)$, we set $p_k = 1,\ a_k = 0$.

Definition. *We shall call the sequence*

$$(p_1, q_1),\ (p_2, q_2),\ \cdots,\ (p_h, q_h)$$

the (p, q)-sequence *of* C *(or of* f*).*

By Lemma 3, we have

Proposition 3 $p_k q_k - b_k p_k - a_k q_k = \begin{cases} -1 & (k = 1) \\ 1 & (k > 1) \end{cases}$

Therefore, $(p_k,\ q_k)$ are coprime each other, and if the pair $(p_k,\ q_k)$ is given, the pair $(a_k,\ b_k)$ $(0 < a_k < p_k,\ 0 < b_k < q_k)$ is determined uniquely by this equation, so that, by the continuous fraction expansion of $\dfrac{p_k}{a_k}$ and $\dfrac{q_k}{b_k}$, the self-intersection numbers of the irreducible components of L_k are determined except for that of E_{i_k}.

Note that $(E_T^2) = (E_{i_h}^2) = -1$. As for the self-intersection numbers of E_{i_1}, $\cdots$, $E_{i_{h-1}}$, one can check easily the following proposition.

Proposition 4 *Suppose that, for $1 < k \leq h$, the t weights $n_{s-t+1}, \cdots, n_s$ of the above dual graph $\Gamma(L_k)$ are equal to -2 and that $n_{s-t} \neq -2$. Then,*

$$(E_{i_{k-1}}^2) = \begin{cases} 2 & (p_k > q_k) \\ 2+t & (p_k = 1) \\ 3+t & (otherwise). \end{cases}$$

Thus, the dual graph $\Gamma(E) = \Gamma(E_C)$ can be determined completely by the (p,q)-sequence of C.

Now, by Lemma 1,

Proposition 5 *The order of the pole of f on E_{i_k} is equal to $q_k\delta_k$.*

In particular, since $\delta_0 = n, \delta_1 = m$, f has the pole of order $p_1 n = q_1 m$ on E_{i_1}. Remember that $m = pd$, $n = qd$, $(p,q) = 1$ (see (2.1)). We get then $p_1 q = q_1 p$. Since $(p_1, q_1) = 1$ by Proposition 3, this implies

Corollary. (i) $p_1 = p,\ q_1 = q$, (ii) $dp_1 = \delta_1$, (iii)$q_1\delta_1 = p_1\delta_0$.

Let

$$P_f = \sum_{i=0}^{T} \nu_i E_i$$

be the pole divisor of f on M. By Lemma 1, the coefficients ν_i of E_i corresponding to the vertices between E_0 (resp. E_1) and E_{i_1}, including E_{i_1}, are all multiple of δ_0 (resp. δ_1). Further, applying

$$(P_f \cdot E_i) = 0$$

for each E_i corresponding to the vertices between E_{i_1} and E_{i_2} successively, all the coefficients ν_i of these E_i, including E_{i_2}, are linear combinations of δ_0 and δ_1 with integer coefficients. Since all the coefficients of E_i between E_{i_2} and E_{j_2} are multiple of δ_2, the coefficients of E_i between E_{i_2} and E_{i_3} are linear combinations of $\delta_0,\ \delta_1,\ \delta_2$ with integer coefficients. Continuing in this way, we see that

Proposition 6 $q_k\delta_k \in \boldsymbol{Z}\delta_0 + \boldsymbol{Z}\delta_1 + \cdots \boldsymbol{Z}\delta_{k-1}$.

Now, setting $\sigma = i_{k-1}$ $(2 \leq k \leq h)$, let us denote by M_σ the surface obtained by the $(\sigma - 1)$-th blowing up in the process to get M from M_1. We may say that M_σ is the surface obtained by the blowing down of $E_T,\ T_{T-1},\ ...,\ E_{\sigma+1}$ successively from M. Let $\tau_\sigma : M \to M_\sigma$ be the mapping obtained by the composition of these blowing downs. We shall denote the images of $\overline{C},\ E_\sigma$ in M_σ by $\overline{C}^{(\sigma)},\ E_\sigma^{(\sigma)}$, respectively. $\overline{C}^{(\sigma)}$ intersects $E_\sigma^{(\sigma)}$ at the point $Q_\sigma = \tau_\sigma(E_T \cup \cdots E_{\sigma+1})$. Let d_k be the intersection number of $\overline{C}^{(\sigma)}$ and $E_\sigma^{(\sigma)}$ in M_σ. Since M_σ is non-singular, there exists a holomorphic function φ_σ on a neighborhood U_σ of $\tau_\sigma^{-1}(Q_\sigma) = E_{\sigma+1} \cup \cdots \cup E_T$ in M which has the zero divisor of the form:

$$(\varphi_\sigma) = E_\sigma \cap U_\sigma + \sum_{i=\sigma+1}^{T} \mu_i E_i.$$

We have then $\mu_{j_\alpha} = \mu_{i_{\alpha-1}}$ and, by lemma 1, $\mu_{i_\alpha} = q_\alpha \mu_{j_\alpha}$ for $\alpha = k, k+1, \cdots, h$, where we set $\mu_\sigma = 1$. This implies $\mu_T = \mu_{i_h} = q_k q_{k+1} \cdots q_h$. Hence, we have

$$d_k = (\overline{C}^{(\sigma)} \cdot E_\sigma^{(\sigma)}) = \mu_T = \begin{cases} 1 & (k = h+1) \\ q_k q_{k+1} \cdots q_h & (k \leq h). \end{cases}$$

Let $P_f^{(\sigma)}$ be the pole divisor of f on M_σ. We have then

$$\overline{C}^{(\sigma)} \sim P_f^{(\sigma)} = \sum_{i \leq \sigma} \nu_i E_i^{(\sigma)},$$

$$d_k = (\overline{C}^{(\sigma)} \cdot E_\sigma^{(\sigma)}) = \sum_{i \leq \sigma} \nu_i (E_i^{(\sigma)} \cdot E_\sigma^{(\sigma)}).$$

Since each ν_i is a linear combination of $\delta_0,\ \delta_1,\ \cdots,\ \delta_{k-1}$ with integer coefficients, we have

$$d_k \in \boldsymbol{Z}\delta_0 + \boldsymbol{Z}\delta_1 + \cdots \boldsymbol{Z}\delta_{k-1}.$$

Hence, d_k is a multiple of $\gcd\{\delta_0, \delta_1, \cdots, \delta_{k-1}\}$. On the other hand, the coefficients $\nu_0, \cdots, \nu_\sigma$ of $P_f^{(\sigma)}$ are the solutions of the simultaneous linear equations

$$\sum_{i \leq \sigma} \nu_i (E_i^{(\sigma)} \cdot E_j^{(\sigma)}) = \begin{cases} 0 & (j < \sigma) \\ d_k & (j = \sigma). \end{cases}$$

Therefore, by Lemma 4, $\nu_0,\ \cdots,\ \nu_\sigma$ are all multiple of d_k. In particular, $\delta_0,\ \delta_1,\ \cdots,\ \delta_{k-1}$ are also multiple of d_k. Thus, d_k is the greatest common divisor of $\delta_0,\ \delta_1,\ \cdots,\ \delta_{k-1}$. Consequently, we have

Proposition 7 *Let d_k $(1 \leq k \leq h+1)$ be the greatest common divisor of $\delta_0,\ \delta_1,\ \cdots,\ \delta_{k-1}$. We have then $d_{h+1} = 1$ and, for $k \leq h$,*

$$d_k = q_k q_{k+1} \cdots q_h,$$

or equivalently

$$q_k = d_k / d_{k+1}.$$

Remark. These relations hold for k=1 also, since $d_1 = \delta_0 = n$, $d_2 = d$ and $q_1 = q = n/d$.

Let us consider the holomorphic function φ_σ on a neighborhood U_σ of $E_{\sigma+1} \cup \cdots \cup E_T$ defined in the above proof of Proposition 7 ($\sigma = i_{k-1}, 2 \le k \le h$). Then, $\Phi = \varphi_\sigma^{\delta_k} f$ takes a pole of order $q_{k-1}\delta_{k-1} - \delta_k$ on $E_{i_{k-1}}$, since f takes a pole of order $q_{k-1}\delta_{k-1}$ on it.

On the other hand, φ_σ takes a zero of order q_k on E_{i_k}, while f takes a pole of order $q_k\delta_k$. Hence, $\Phi = \varphi_\sigma^{\delta_k} f$ is either a non-constant function or a non-zero constant ($\ne \infty$) on E_{i_k}. If one blow down $E_T, E_{T-1}, \cdots, E_{i_k+1}$, the zero curve $\overline{C}^{(i_k)}$ of Φ intersects $E_{i_k}^{(i_k)}$ with the multiplicity $d_{k+1} = q_{k+1}q_{k+2}\cdots q_h$. Therefore Φ is non-constnt and of degree d_{k+1} on E_{i_k}. Hence, the order of the pole of Φ on $E_{i_{k-1}}$ is $d_{k+1}p_k$. Consequently, we obtain

Proposition 8 $d_{k+1}p_k = q_{k-1}\delta_{k-1} - \delta_k$ $(2 \le k \le h)$.

Thus, the (p, q)-sequence and the dual graph $\Gamma(E)$ are determined completely by the δ-sequence of f. Let us summarize here the results obtained so far about the δ-sequence and (p, q)-sequence.

Proposition 9 *Let $C : f = 0$ be a non-linearizable affine plane curve with one place at infinity. Let $\delta_0, \delta_1, \cdots, \delta_h$ be the δ-sequence of C, $\{(p_k, q_k)\}$ be the (p, q)-sequence of C and set $d_k = \gcd\{\delta_0, \cdots, \delta_{k-1}\}$ $(1 \le k \le h + 1)$. We have then, for $1 \le k \le h$,*

(1) $q_k = d_k/d_{k+1}$, $d_{h+1} = 1$,

(2) $d_{k+1}p_k = \begin{cases} \delta_1 & (k = 1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \le k \le h), \end{cases}$

(3) $q_k\delta_k \in \boldsymbol{Z}\delta_0 + \boldsymbol{Z}\delta_1 + \cdots + \boldsymbol{Z}\delta_{k-1}$.

Further, the dual graph $\Gamma(E_C)$ of the minimal resolution of the singularity of C at infinity is determined by the δ-sequence.

4 Canonical Divisor

The holomorphic 2-form $\omega = dx \wedge dy$ in $\boldsymbol{C}^2$ extends to a meromorphic 2-form on M. The canonical divisor $K = (\omega)$ has the support on E. Let g be the genus of the curve $C_\alpha : f = \alpha$ for generic $\alpha \in \boldsymbol{C}$. If C is non-singular, g is equal to the genus of C by Theorem 1. Let

$$P_f = \sum_{i=0}^{T} \nu_i E_i$$

be the pole divisor of f on M. Taking into account

$$(P_f \cdot E_i) = (\overline{C} \cdot E_i) = \begin{cases} 0 & (0 \le i \le T - 1) \\ 1 & (i = T), \end{cases}$$

we get, by the adjunction formula,

$$\begin{aligned}
2g-2 &= (K\cdot\overline{C}) + (\overline{C}^2) \\
&= (K\cdot P_f) + (\overline{C}\cdot P_f) \\
&= (K\cdot P_f) + \nu_T \\
&= ((K+E)\cdot P_f) - (E_T\cdot P_f) + \nu_T \\
&= \sum_{i=0}^{T} \nu_i((K+E)\cdot E_i) - 1 + \nu_T
\end{aligned}$$

According as E_i corresponds either the branch point, edge, or other point in the dual graph $\Gamma(E)$, we can calculate the value of $(K+E)\cdot E_i$ as follows:

$$(K+E)\cdot E_i = \begin{cases} 1 & \text{(branch)} \\ -1 & \text{(edge)} \\ 0 & \text{(others).} \end{cases}$$

Hence

$$\begin{aligned}
2g-2 &= \nu_{i_1} - \nu_0 - \nu_1 + \sum_{k=2}^{h-1}(\nu_{i_k} - \nu_{j_k}) - \nu_{j_h} + \nu_T - 1 \\
&= \nu_{i_1} - \nu_0 - \nu_1 + \sum_{k=2}^{h}(\nu_{i_k} - \nu_{j_k}) - 1 \\
&= pqd - pd - qd - 1 + \sum_{k=2}^{h}\delta_k(q_k - 1).
\end{aligned}$$

Thus, we have

Theorem 2 $2g-1 = d\{(p-1)(q-1)-1\} + \sum_{k=2}^{h}\delta_k(q_k-1).$

Since the right hand side of this last equation is positive, we have $g > 0$. We get in this way another proof to the Corollary 3 of Theorem 1.

5 Approximate roots

Multiplying $f(x,y)$ by a non-zero constant, if necessary, we may assume that $f(x,y)$ is *monic* (of degree n) with respect to y. For the divisors $d_1(=n)$, d_2, $\cdots$, d_h, $d_{h+1}(=1)$ of n defined in Proposition 7, set $n_k = \dfrac{n}{d_k}$ $(k = 1,\cdots,h+1)$. Then, there exists, for each k $(1\le k\le h+1)$, a pair of polynomials $g_k(x,y)$, monic and of degree n_k with respect to y, and $\psi_k(x,y)$ of degree $< n-n_k$ with respect to y, uniquely determined by the folowing condition:

$$f = g_k^{d_k} + \psi_k.$$

One can check the existence and the uniqueness of this pair (g_k, ψ_k) by the termwise comparison of the both side of the last equation. We shall call $g_k(x,y)$ the *k-th approximate root* of f. We have $g_1 = y$ and $g_{h+1} = f$ by definition. From the uniqueness of the approximate roots, it follows that g_k is also the k-th approximate root of g_j for any j with $k < j < h+1$. The sequence

$$g_0 = x,\ g_1 = y,\ g_2,\ \cdots,\ g_h,\ g_{h+1} = f$$

will be called the *g-sequence* of f. In the followings, we shall denote by C_k the curve defined by $g_k(x,y) = 0$ in $\boldsymbol{C}^2$. We have $C_0 = y$-axis, $C_1 = x$-axis and $C = C_{h+1}$ by definition.

Theorem 3 *Each C_k $(k \leq h)$ is also with one place at infinity. Further, its closure $\overline{C_k}$ in M intersects transversely E_{j_k}, and does not intersect other irreducible components of E.*

Before giving the proof to this theorem, let us prepare two lemmas.

Lemma 5 *Let $c_1,\ c_2, \cdots,\ c_\alpha$ be α non-zero complex numbers, d an integer, and set*

$$\varphi(t) = \prod_{i=1}^{\alpha} (t - c_i)^d.$$

Then, $\varphi(t) - \varphi(0)$ takes a zeros of order $\leq \alpha$ at $t = 0$.

Lemma 6 *Let $a,\ b,\ e$ be complex analytic curves on a non-singular complex surface W such that a and b have no common irreducible component with e, $a \cap b = \emptyset$ and e is compact. Assume that $a \cup e$ (resp. $b \cup e$) be the zero set of a holomorphic function u (resp. v) on W. Let*

$$a = \bigcup_{i=1}^{r} a_i,\ b = \bigcup_{j=1}^{s} b_j,\ e = \bigcup_{k=1}^{t} e_k$$

be the decompositions into the irreducible components, and let

$$(u) = \sum_{i=1}^{r} \mu_i a_i + \sum_{k=1}^{t} m_k e_k,$$

$$(v) = \sum_{j=1}^{s} \nu_j b_j + \sum_{k=1}^{t} n_k e_k$$

be the zero divisor of u, v respectively. Finaly, let α_i (resp. β_j) be the degree of the zero divisor of v (resp. u) restricted to a_i (resp. b_j). We have then,

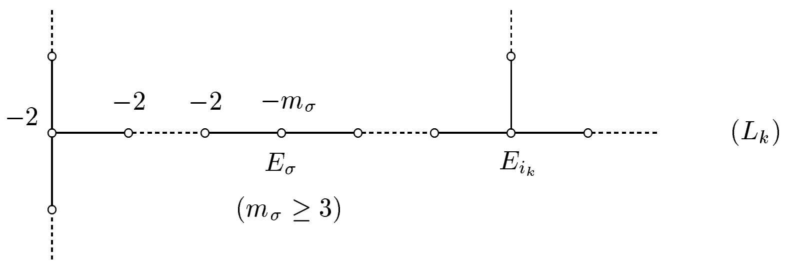
$$((u)\cdot(v)) = \sum_{i=1}^{r} \mu_i \alpha_i = \sum_{j=1}^{s} \nu_j \beta_j.$$

In fact, since $a_i \cdot b_j = 0,\ (u)\cdot e_k = 0,\ (v)\cdot e_k = 0,$

$$\begin{aligned}
((u)\cdot(v)) &= (\sum_{i=1}^{r} \mu_i a_i + \sum_{k=1}^{t} m_k e_k)\cdot(v)) \\
&= (\sum_{i=1}^{r} \mu_i a_i)\cdot(v)) \\
&= \sum_{i=1}^{r} \mu_i (a_i \cdot (v)) \\
&= \sum_{i=1}^{r} \mu_i \alpha_i.
\end{aligned}$$

In the same way, we have $((u)\cdot(v)) = \sum_{j=1}^{s} \nu_j \beta_j$.

Proof of Theorem 3 It is sufficient to prove it for $2 \le k \le h$. Set $\sigma = j_k - 1$. In the case $(E^2_{i_{k-1}}) \le -3$, we have $\sigma = i_{k-1}$. In the other case $(E^2_{i_{k-1}}) = -2$, E_σ is the component with the self-intersection number ≤ -3 on L_k nearest to $E_{i_{k-1}}$ on the dual graph $\Gamma(E)$. (See the figure below.)



Let M_σ be the surface obtained by the $(\sigma - 1)$-th blowing up in the process to get M from M_1. We may say that M_σ is the surface obtained by the blowing down of $E_T,\ E_{T-1},\ \ldots,\ E_{\sigma+1}$ successively from M. We shall denote the images of $\overline{C},\ \overline{C}_k,\ E_i$ in M_σ by $\overline{C}^{(\sigma)},\ \overline{C}_k^{(\sigma)},\ E_i^{(\sigma)}$, respectively. Let Q be the intersection point of $\overline{C}^{(\sigma)}$ and $E_\sigma^{(\sigma)}$. Note that we have

$$(\overline{C}^{(\sigma)} \cdot E_\sigma^{(\sigma)}) = d_k,$$

and $\overline{C}^{(\sigma)}$ is tangent to $E_\sigma^{(\sigma)}$, since $({E_\sigma}^2) \leq -3$.

Now, suppose that $\overline{C}_k^{(\sigma)}$ passes through the point Q. Let $P_x^{(\sigma)} = \sum_{i=1}^{\sigma} \mu_i E_i^{(\sigma)}$ be the pole divisor of x on M_σ. We have $\mu_i > 0$ for $1 \leq i \leq \sigma$. We have

$$\begin{aligned} n_k &= \text{the } y\text{-degree of the function } g_k \\ &= \text{the intersection number of } C_k \text{ and the line } x = \text{const.} \\ &= \text{the degree of } x \text{ on } C_k \\ &= (\overline{C}_k^{(\sigma)} \cdot P_x^{(\sigma)}) = \sum_{i=1}^{\sigma} \mu_i (\overline{C}_k^{(\sigma)} \cdot E_i^{(\sigma)}). \end{aligned}$$

On the other hand, as one can check it easily using Lemma 1, the coefficient μ_σ, the order of the pole of x on E_σ, is equal to $q_1 \cdot q_2 \cdots q_{k-1} = n_k$. Hence, we have

$$(\overline{C}_k^{(\sigma)} \cdot E_i^{(\sigma)}) = \begin{cases} 0 & (1 \leq i \leq \sigma - 1) \\ 1 & (i = \sigma), \end{cases}$$

while $(\overline{C}_k^{(\sigma)} \cdot E_0^{(\sigma)}) = 0$, since g_k is monic. Thus, C_k has one place at infinity. Since $\overline{C}_k^{(\sigma)}$ intersects $E_\sigma^{(\sigma)}$ transversely at Q, $\overline{C}_k$ intersects $E_{\sigma+1} = E_{j_k}$ transeversely at the regular point of E in M. Thus, if one shows that $\overline{C}_k^{(\sigma)}$ passes through the point Q, the proof of the Theorem 3 will be accomplished.

Suppose that $\overline{C}_k^{(\sigma)}$ does not pass through the point Q. Consider the rational function

$$\Phi = \frac{g_k^{d_k}}{f} = 1 - \frac{\psi_k}{f}$$

on M_σ. Φ has no zero in $U - E_\sigma^{(\sigma)} \cap U$ for a small neighborhood U of Q. Assume that Q is an indetermination point of Φ. Then, there must be a zero curve of Φ which passes through Q. So, $\Phi = 0$ on $E_\sigma^{(\sigma)}$. Set $A^{(\sigma)} = E_0^{(\sigma)} \cup E_1^{(\sigma)} \cup \cdots \cup E_{\sigma-1}^{(\sigma)}$. Since $A^{(\sigma)}$ is exceptional and $A^{(\sigma)} \cap \overline{C}^{(\sigma)} = \emptyset$, Φ has no pole on $A^{(\sigma)}$. Therefore, Φ must be constant $(= 0)$ on $A^{(\sigma)}$. But, since $\deg_y \psi_k < n - n_k$, we have $\psi_k / f = 0$ on $E_0^{(\sigma)}$, so that $\Phi = 1$ on $E_0^{(\sigma)}$. This is a contradiction. Hence, Q is not an indetermination point of Φ.

Now, blow up the indetermination points of Φ until the indetermination points of Φ disappear. Let $\tilde{M}$ be the surface thus obtained. We shall denote by $\tilde{C}$, $\tilde{C}_k$, $\tilde{E}_j$, etc. the proper images in $\tilde{M}$ of $\overline{C}^{(\sigma)}$, $\overline{C}_k^{(\sigma)}$, $E_j^{(\sigma)}$, etc. respectively. In $\tilde{M}$, we can write the divisor of Φ as follows:

$$(\Phi) = d_k \tilde{C}_k - \tilde{C} + \sum_{j=0}^{\tau} \nu_j \tilde{E}_j,$$

namely,

$$\tilde{C} \sim d_k \tilde{C}_k + \sum_{j=0}^{\tau} \nu_j \tilde{E}_j ,$$

where $\tilde{E}_0, \tilde{E}_1, \cdots, \tilde{E}_\sigma$ are the proper images of $E_0^{(\sigma)}, E_1^{(\sigma)}, \cdots, E_\sigma^{(\sigma)}$ respectively, and $\tilde{E}_{\sigma+1}, \cdots, \tilde{E}_\tau$ are the curves appeared by the blowing ups. Since Q is not an indetermination point, we have

$$(\tilde{C} \cdot \tilde{E}_i) = \begin{cases} 0 & (i \neq \sigma) \\ d_k & (i = \sigma), \end{cases}$$

Hence, $\sum_{j=0}^{\tau} \nu_j (\tilde{E}_j \cdot E_i)$ $(0 \leq i \leq \tau)$ are all multiple of d_k, so that, by Lemma 4, all the coefficients ν_j are multiple of d_k.

Now, if the pole set $\tilde{P}$ of Φ on $\tilde{E}$ is not connected, then one of its connected component must be included in $\tilde{A}$. But this is a contradiction, since $\tilde{C} \cap \tilde{A} = \emptyset$ and $\tilde{A}$ is exceptional. Therefore, $\tilde{P}$ is connected.

Let $\{\tilde{E}_\lambda\}_{\lambda \in \Lambda}$ be the irreducible components of $\tilde{E}$ on which Φ is non-constant. Take one $\tilde{E}_\lambda$ $(\lambda \in \Lambda)$. Since the dual graph of $\tilde{E}$ is a tree and $\tilde{P}$ is connected, Φ has the pole on $\tilde{E}_\lambda$ at only one point, say a. Let $\tilde{B}_0$ be the connected component of the closure of $\tilde{E} - \tilde{E}_\lambda$ in $\tilde{M}$ which contains E_0. Then, $\Phi = 1$ on $\tilde{B}_0$, since Φ is holomorphic in a neighborhood of $\tilde{B}_0$ and $\Phi = 1$ on $E_0^{(\sigma)}$. Let b be the intersection point of $\tilde{B}_0$ and $\tilde{E}_\lambda$, and take a coordinate function t of $\tilde{E}_\lambda \cong \boldsymbol{P}^1 : |t| \leq \infty$ such that $t(a) = \infty$ and $t(b) = 0$. Since all the ν_j are multiple of d_k, we can write $\varphi = \Phi|_{\tilde{E}_\lambda}$ as follows:

$$\varphi(t) = \prod_{i=1}^{\alpha_\lambda} (t - c_i)^{d_k}.$$

By Lemma 5, the number of the zeros of $\varphi(t) - 1$ $(= -\psi_k / f)$ other than $t = 0$ is $\geq \alpha_\lambda (d_k - 1)$.

Let $\tilde{P}_x = \sum_{i=0}^{\tau} \mu_i \tilde{E}_i^{(\sigma)}$ be the pole divisor of x on $\tilde{M}$ and $\tilde{Z}_\Phi$ the zero divisor of Φ. The support of $\tilde{Z}_\Phi$ does not intersect $\tilde{B}_0$. We have, by Lemma 6,

$$\begin{aligned} (\tilde{P}_x \cdot d_k \tilde{C}_k) &= (\tilde{P}_x \cdot \tilde{Z}_\Phi) \\ &= \sum_{\lambda \in \Lambda} \mu_\lambda d_k \alpha_\lambda , \end{aligned}$$

so that

$$n_k = \deg_y (g_k) = (\tilde{P}_x \cdot \tilde{C}_k) = \sum_{\lambda \in \Lambda} \mu_\lambda \alpha_\lambda .$$

In the same way, let Σ_k be the curve defined by $\psi_k = 0$ in $\boldsymbol{C}^2$ and $\tilde{\Sigma}_k$ its closure in $\tilde{M}$. We have then, by Lemma 6,

$$\deg_y(\psi_k) = (\tilde{P}_x \cdot \tilde{\Sigma}_k) \geq \sum_{\lambda \in \Lambda} \mu_\lambda \alpha_\lambda (d_k - 1).$$

Hence,

$$\deg_y(\psi_k) \geq n_k(d_k - 1) = n - n_k.$$

This is contrary to the assumption $\deg_y(\psi_k) < n - n_k$. Hence, $\overline{C}_k^{(\sigma)}$ passes through the point Q, and theorem 3 is proved.

Corollary. *Each g_k $(0 \leq k \leq h)$ has the pole of order δ_k on $E_T = E_{i_h}$.*

Infact, let P_f (resp. P_{g_k}) be the pole divisor of f (resp. g_k) on M. Then, we have, by Theorem 3,

$$\begin{aligned} \delta_k &= (P_f \cdot \overline{C}_k) \\ &= (\overline{C} \cdot \overline{C}_k) \\ &= (\overline{C} \cdot P_{g_k}) \\ &= \text{the order of the pole of } g_k \text{ on } E_T. \end{aligned}$$

6 Semigroup Critrion

We shall continue to use the same notations as in the previous sections.

Lemma 7 *For any integer k $(1 \leq k \leq h)$ and any integer λ, the sequence of integers $\{\alpha_0, \alpha_1, \cdots, \alpha_k\}$ satisfying*

$$\lambda = \alpha_0\delta_0 + \alpha_1\delta_1 + \cdots + \alpha_k\delta_k$$

and $0 \leq \alpha_i < q_i$ for $1 \leq i \leq k$ is unique, if it does exists.

In fact, assume that there exist two sequences $\{\alpha_i\}$, $\{\beta_i\}$ with $0 \leq \alpha_i, \beta_i < q_i$ $(1 \leq i \leq k)$ satisfying

$$(1) \qquad \sum_{i=0}^{k} \alpha_i\delta_i = \sum_{i=0}^{k} \beta_i\delta_i.$$

Then, setting $\bar{\delta}_i = \delta_i/d_{k+1}$, we have

$$\sum_{i=0}^{k-1} (\alpha_i - \beta_i)\bar{\delta}_i = (\beta_k - \alpha_k)\bar{\delta}_k.$$

Here, $\bar{\delta}_0, \cdots, \bar{\delta}_{k-1}$ and $\bar{\delta}_k$ are mutually coprime, since $d_{k+1} = \gcd\{\delta_0, \cdots, \delta_k\}$. Therefore, $\beta_k - \alpha_k$ must be a multiple of $\gcd\{\bar{\delta}_0, \cdots, \bar{\delta}_{k-1}\} = d_k/d_{k+1} = q_k$. Since $0 \leq \alpha_k, \beta_k < q_k$, this implies $\alpha_k = \beta_k$. Repeating the same argument, we get $\alpha_{k-1} = \beta_{k-1}, \cdots, \alpha_1 = \beta_1$, and finally, $\alpha_0 = \beta_0$ by the equation (1).

Proposition 10 *For each k $(1 \le k \le h)$, $q_k\delta_k$ is a linear combination of δ_0, $\delta_1, \cdots, \delta_{k-1}$ with non-negative integer coefficients, namely*

$$q_k\delta_k \in \boldsymbol{N}\delta_0 + \boldsymbol{N}\delta_1 + \cdots + \boldsymbol{N}\delta_{k-1}. \tag{2}$$

Proof. We have $q_1\delta_1 = p\delta_0$ by the corollary to Proposition 5. Therefore, it is sufficient to prove (2) for $k \ge 2$. Set $\sigma = i_k$, and let us consider the surface M_σ obtained by the $(\sigma-1)$-th blowing up in the process to get M from M_1. We may say that M_σ is the surface obtained by the blowing down of L_{h+1}, L_h, ..., L_{k+1} successively from M. Let $\pi_\sigma : M \to M_\sigma$ be the contraction mapping. As in the previous sections, let us denote the proper images of $\overline{C}$, $\overline{C}_k$, E_i in M_σ by $\overline{C}^{(\sigma)}$, $\overline{C}_k^{(\sigma)}$, $E_i^{(\sigma)}$ respectively. By Theorem 3, $\overline{C}_{k+1}^{(\sigma)}$ intersects transversely $E_\sigma^{(\sigma)}$ at the same point $Q = \pi_\sigma(L_{k+1} \cup \cdots \cup L_{h+1})$ as $\overline{C}^{(\sigma)}$. Hence, the functions f and g_{k+1} on M_σ have the same indetermination point $Q \in E_\sigma^{(\sigma)}$. Let

$$P_f^{(\sigma)} = \sum_{i=0}^{\sigma} \nu_i E_i^{(\sigma)}, \quad P_{g_{k+1}}^{(\sigma)} = \sum_{i=0}^{\sigma} \bar{\nu}_i E_i^{(\sigma)}$$

be the pole divisor of f and g_{k+1} on M_σ respectively. Let $\bar{\delta}_0, \bar{\delta}_1, \cdots, \bar{\delta}_k$ be the order of the pole of g_{k+1} on $E_{j_0}(=E_0)$, $E_{j_1}(=E_1)$, $\cdots$, E_{j_k}. We have $\bar{\delta}_0 = \bar{\nu}_{j_0}$, $\bar{\delta}_1 = \bar{\nu}_{j_1}, \cdots, \bar{\delta}_k = \bar{\nu}_{j_k}$. The coefficients ν_i, $\bar{\nu}_i$ $(i = 0, 1, \cdots, \sigma)$ are the solutions of the following equations:

$$\begin{aligned} \sum_{j=0}^{\sigma}(E_i^{(\sigma)} \cdot E_j^{(\sigma)})\nu_j &= \begin{cases} 0 & (i \ne \sigma) \\ d_{k+1} & (i = \sigma), \end{cases} \\ \sum_{j=0}^{\sigma}(E_i^{(\sigma)} \cdot E_j^{(\sigma)})\bar{\nu}_j &= \begin{cases} 0 & (i \ne \sigma) \\ 1 & (i = \sigma). \end{cases} \end{aligned}$$

Hence, by Lemma 4, we have $\nu_i = d_{k+1}\bar{\nu}_i$ for all $i = 0, 1, \cdots, \sigma$. In particular,

$$\delta_i = \bar{\delta}_i \cdot d_{k+1}, \quad (i = 0, 1, \cdots, k).$$

Therefore, in order to prove (2), it is sufficient to prove

$$q_k\bar{\delta}_k \in \boldsymbol{N}\bar{\delta}_0 + \boldsymbol{N}\bar{\delta}_1 + \cdots + \boldsymbol{N}\bar{\delta}_{k-1}. \tag{3}$$

By Theorem 3, $\overline{C}_k^{(\sigma)}$ intersects $E_{j_k}^{(\sigma)}$ transversely and does not intersects other components $E_i^{(\sigma)}$ $(i \ne j_k)$. We have

$$\begin{aligned} \bar{\delta}_k &= (P_{g_{k+1}}^{(\sigma)} \cdot \overline{C}_k^{(\sigma)}) \\ &= (\overline{C}_{k+1}^{(\sigma)} \cdot \overline{C}_k^{(\sigma)}) \\ &= (\overline{C}_{k+1}^{(\sigma)} \cdot P_{g_k}^{(\sigma)}). \end{aligned}$$

This implies that g_k has the pole of order $\bar{\delta}_k$ on $E_\sigma^{(\sigma)}$. On the other hand, by Lemma 1, g_{k+1} has the pole of order $q_k\bar{\delta}_k$ on $E_\sigma^{(\sigma)}$. Hence, $E_\sigma^{(\sigma)}$ is neither the zero nor the pole of $\Phi = \dfrac{g_{k+1}}{g_k^{q_k}}$. Further, Φ is holomorphic in a neighborhood of Q and $\Phi(Q) = 0$. Therefore, Φ is not constant on $E_\sigma^{(\sigma)}$.

Now, set $\psi = g_{k+1} - g_k^{q_k}$. Then,

$$\frac{\psi}{g_k^{q_k}} = \Phi - 1$$

is also a non-constant function on $E_\sigma^{(\sigma)}$. Therefore, ψ has also the pole of order $q_k\bar{\delta}_k$ on $E_\sigma^{(\sigma)}$. On the other hand, since

$$\deg_y(\psi) < n_{k+1} = n_k q_k, \quad n_k = \deg_y(g_k),$$

by the division of ψ by $g_k^{q_k-1}$, we get

$$\psi = c_1 g_k^{q_k-1} + \psi_1$$

with $\deg_y(c_1) < n_k,\ \deg_y(\psi_1) < n_k(q_k - 1)$. Dividing ψ_{i-1} by $g_k^{q_k-i}$ successively for $i = 2, \cdots, q_k - 1$, we get

$$\psi_{i-1} = c_i g_k^{q_k-i} + \psi_i,$$

where $\deg_y(c_1) < n_k,\ \deg_y(\psi_i) < n_k(q_k - i)$. Thus, setting $c_{q_k} = \psi_{q_k-1}$, we get

$$\psi = \sum_{i=1}^{q_k} c_i g_k^{q_k-i}.$$

Here, we have

$$\deg_y(c_i) < n_k = n_{k-1}q_{k-1}, \quad n_{k-1} = \deg_y(g_{k-1}).$$

In the same way, dividing c_i and its rests by $g_{k-1}^{q_{k-1}-1}$, $g_{k-1}^{q_{k-1}-2}$, $\cdots$, g_{k-1} successively, we get

$$c_i = \sum_{j=1}^{q_{k-1}} c_{ij} g_{k-1}^{q_{k-1}-j}$$

with $\deg_y(c_{ij}) < n_{k-1}$. Thus, we have

$$\psi = \sum_{i=1}^{q_k} \sum_{j=1}^{q_{k-1}} c_{ij} g_k^{q_k-i} g_{k-1}^{q_{k-1}-j}.$$

Repeating this procedure, we obtain

$$\psi = \sum_{\alpha_1 < q_1, \alpha_2 < q_2, \cdots, \alpha_k < q_k} c_{\alpha_1\alpha_2\cdots\alpha_k} g_1^{\alpha_1} g_2^{\alpha_2} \cdots g_k^{\alpha_k},$$

where $\deg_y(c_{\alpha_1\alpha_2\cdots\alpha_k}) = 0$, since $g_1 = y$. Substituting finally $g_0 = x$, we can write ψ as follows:

$$\psi = \sum_{\alpha_0<\infty,\alpha_1<q_1,\cdots,\alpha_k<q_k} c_{\alpha_0\alpha_1\cdots\alpha_k} g_0^{\alpha_0} g_1^{\alpha_1}\cdots g_k^{\alpha_k},$$

where $c_{\alpha_0\alpha_1\cdots\alpha_k}$ are constants.

The order of the pole of each term $g_0^{\alpha_0} g_1^{\alpha_1}\cdots g_k^{\alpha_k}$ on $E_\sigma^{(\sigma)}$ is

$$\alpha_0\delta_0 + \alpha_1\delta_1 + \cdots + \alpha_k\delta_k. \tag{4}$$

Now, by Lemma 7, these values are different each other. Hence, the order of the pole of ψ on $E_\sigma^{(\sigma)}$ coincide with one of the values of (4). Thus, we have

$$q_k\delta_k = \alpha_0\delta_0 + \alpha_1\delta_1 + \cdots + \alpha_k\delta_k$$

for some sequence of non-negative integers $\alpha_0, \alpha_1, \cdots, \alpha_k$ satisfying $0 \le \alpha_i < q_i$ for $i = 1, \cdots, k$. Assume here that $\alpha_k \neq 0$. Then, we have

$$\alpha_0\delta_0 + \alpha_1\delta_1 + \cdots + \alpha_{k-1}\delta_{k-1} + (q_k - \alpha_k)\delta_k = 0$$

with $0 < q_k - \alpha_k < q_k$. This contradicts Lemma 7. Hence, we have $\alpha_k = 0$. Thus, we obtain

$$q_k\delta_k = \alpha_0\delta_0 + \alpha_1\delta_1 + \cdots + \alpha_{k-1}\delta_{k-1}.$$

Q.E.D.

By Proposition 9 and Proposition 10, we obtain the following theorem which can be regarded as an algebrico-geometric version of the so-called semi-group theorem due to Abhyankar and Moh[2] for the curves with one place at infinity in $\boldsymbol{C}^2$.

Theorem 4 *Let C be an affine plane curve with one place at infinity in $\boldsymbol{C}^2$, $\delta_0, \delta_1, \cdots, \delta_h$ the δ-sequence of C and $(p_1, q_1), \cdots, (p_h, q_h)$ the (p,q)-sequence of C defined at the beginning of the section 3. Set $d_k = \gcd\{\delta_0, \cdots, \delta_{k-1}\}$ for $1 \le k \le h+1$. Then, we have, for $1 \le k \le h$,*

(1) $q_k = d_k/d_{k+1},\ d_{h+1} = 1,$

(2) $d_{k+1}p_k = \begin{cases} \delta_0 & (k=1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \le k \le h), \end{cases}$

(3) $q_k\delta_k \in \boldsymbol{N}\delta_0 + \boldsymbol{N}\delta_1 + \cdots \boldsymbol{N}\delta_{k-1}.$

Further, the dual graph $\Gamma(E_C)$ of the minimal resolution of the singularity of C at infinity is determined by the δ-sequence.

The next theorem gives the inverse of Theorem 4 in some sense.

Theorem 5 (Sathaye-Stenerson[10]) *For a sequence of* $h+1$ *natural numbers* $\delta_0,\ \delta_1,\ \delta_2,\ \cdots,\ \delta_h$ $(h \geq 1)$*, define* $d_1,\ d_2,\ \cdots,\ d_{h+1}$ *by*

$$d_k = \gcd\{\delta_0, \cdots, \delta_{k-1}\}$$

and set $q_k = d_k/d_{k+1}$ $(1 \leq k \leq h)$*. Suppose that the following three conditions are satisfied:*

(1) $d_{h+1} = 1$,
(2) $\delta_k < q_{k-1}\delta_{k-1}$ $(2 \leq k \leq h)$,
(3) $q_k\delta_k \in \boldsymbol{N}\delta_0 + \boldsymbol{N}\delta_1 + \cdots + \boldsymbol{N}\delta_{k-1}$ $(1 \leq k \leq h)$.

Then, $\{\delta_0,\ \delta_1,\ \delta_2,\ \cdots,\ \delta_h\}$ *is the* δ*-sequence of an affine plane curve with one place at infinity in* $\boldsymbol{C}^2$.

Proof We shall prove theorem 5 by the induction on h. In the case $h = 1$, setting $\delta_0 = q,\ \delta_1 = p$, we have $(p,\ q) = d_2 = 1$. Hence, as one sees it easily, the curve $x^p + y^q = 0$ has one place at infinity and $\{\delta_0,\ \delta_1\}$ is its δ-sequence.

Now, let us consider the case $h \geq 2$. Set $\tilde{\delta}_i = \delta_k/d_h$ for $0 \leq k \leq h-1$, and $\tilde{d}_k = d_k/d_h$ for $1 \leq k \leq h$. We have

$$\tilde{d}_k = \gcd\{\tilde{\delta}_0, \tilde{\delta}_1, \cdots, \tilde{\delta}_{h-1}\} \text{ and } q_k = \tilde{d}_k/\tilde{d}_{k+1}$$

for $1 \leq k \leq h-1$. Further, the sequence $\{\tilde{\delta}_0, \tilde{\delta}_1, \cdots, \tilde{\delta}_{h-1}\}$ satisfies the same properties (1), (2), (3) for $\tilde{h} = h-1$. Therefore, by the induction hypothesis, there exists an affine plane curve C_h with one place at infinity which has $\{\tilde{\delta}_i\}$ as its δ-sequence. Let $\tilde{f}$ be the defining polynomial of C_h and, taking the canonical coordinate system x, y for C_h (see (2.1)), let $g_0 = x$, $g_1 = y$, $\cdots$, g_{h-1}, $g_h = \tilde{f}$ be its g-sequence. Let $(\tilde{M}, \tilde{E}) = (M_{C_h}, E_{C_h})$ be the compactification of $\boldsymbol{C}^2$ obtained by the minimal resolution of the singularity of C_h at infinity (see (2.3)). The closure $\tilde{C}_k$ of the curve $C_k : g_k = 0$ $(0 \leq k \leq h-1)$ in $\tilde{M}$ passes through the irreducible component $\tilde{E}_{j_k}$ of $\tilde{E}$ corresponding to the k-th edge of the dual graph $\Gamma(\tilde{E})$, and $\tilde{C}_h$ passes through a point Q of $\tilde{E}_{i_{h-1}}$, the curve appeared by the last blowing up. Now, set

$$p_h = (q_{h-1}\delta_{h-1} - \delta_h).$$

Since $q_h = d_h = \gcd\{\delta_0, \cdots, \delta_{h-1}\}$ and $(q_h, \delta_h) = d_{h+1} = 1$, then we have $(p_h, q_h) = 1$. Therefore, we can take $a_h,\ b_h$ $(0 < a_h < p_h,\ 0 < b_h < q_h)$ such that

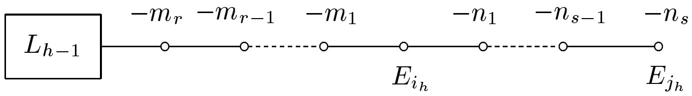
$$p_h q_h - b_h p_h - a_h q_h = 1.$$

Define the integers $m_i, n_j \geq 2$ by the following continuous fraction expansions:

$$p_h/a_h = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \cdots - \cfrac{1}{m_r}}},$$

$$q_h/b_h = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \cdots - \cfrac{1}{n_s}}}.$$

Then, we can blow up the point Q and the infinitely near points successively, in such a way that we get the h-th branch L_h of the form:



Let M be the surface thus obtained and E the total image of $\tilde{E}$ in M. Let E_i, $\overline{C}_k$ the proper images of $\tilde{E}_i$, $\tilde{C}_k$ in M. Note that we can do these blowing up in such a way that the closure $\overline{C}_h$ passes through the component E_{j_h}. (It is sufficient to take the center of the blowing up on E_{j_h} to get E_{j_h+1} outside the intersection of the closure of C_h with E_{j_h}.)

According to the Corollary to Theorem 3, each g_k $(0 \leq k \leq h-1)$ takes the pole of order $\tilde{\delta}_k$ on $E_{i_{h-1}}$, so that it takes the pole of order δ_k on E_{i_h}, while g_h takes the pole of order

$$q_h(q_{h-1}\tilde{\delta}_{h-1}) - p_h = q_{h-1}\delta_{h-1} - p_h = \delta_h$$

on E_{i_h}.

Now, by the condition (3), there exists a sequence of non-negative integers α_i satisfying

$$q_h\delta_h = \alpha_0\delta_0 + \alpha_1\delta_1 + \cdots + \alpha_{h-1}\delta_{h-1},$$

where we may assume $0 \leq \alpha_i \leq q_i$ $(1 \leq i \leq h-1)$, applying the condition (3) for $k = h-1, h-2, \cdots, 1$ successively, if it is neccessary. Consider the polynomial

$$f(x,\ y) = g_h^{q_h} - \prod_{i=0}^{h-1} g_i^{\alpha_i}$$

and the curve $C : f(x,y) = 0$ in $\boldsymbol{C}^2$. Since $g_h^{q_h}$ and $\prod_{i=0}^{h-1} g_i^{\alpha_i}$ have the same order (say $q_h\delta_h$) of the poles on E_{i_h}, the function

$$\Phi = g_h^{-q_h} \prod_{i=0}^{h-1} g_i^{\alpha_i} = 1 - \frac{f}{g_h^{q_h}}$$

is either a non-constant function or a non-zero constant on E_{i_h}.

Let A (resp.B) be the closure of the connected component of $E - E_{i_h}$ which contains E_0 (resp.E_{j_h}). Since A is exceptional and $A \cap \overline{C}_h = \emptyset$, Φ is holomorphic on A and the pole of Φ is contained in $\overline{C}_h \cup B$. On the other hand,

$$\deg_y(\prod_{i=0}^{h-1} g_i^{\alpha_i}) < \sum_{i=1}^{h-1} q_i n_i = \sum_{i=1}^{h-1} q_i q_{i-1} \cdots q_1 < q_h q_{h-1} \cdots q_1 = \deg_y(g_h^{q_h}).$$

Hence, E_0 is a zero of Φ, so that $\Phi = 0$ on A. Therefore, Φ is non-constant on E_{i_h}, and takes the pole on the irreducible component B_1 of B which intersects E_{i_h}. The pole divisor of Φ must be of the form

$$P_\Phi = q_h \overline{C}_h + \sum_{i=1}^{s} \mu_i B_i,\ (\mu_i > 0),$$

where $B_1,\ B_2,\ \cdots, B_s$ are the irreducible components of B. By Lemma 1, we have $q_h \mu_1 = q_h$, so that $\mu_1 = 1$. Thus, Φ is a rational function of degree 1 on E_{i_h}. Since the curve $\Phi = 1$ coincide with $\overline{C}$, we obtain

$$(\overline{C} \cdot E_i) = \begin{cases} 1 & (i = i_h) \\ 0 & (i \neq i_h). \end{cases}$$

Thus, the curve C has one place at infinity.

Now, since $\Phi = 0$ on A, f takes the same order of the pole as $g_h^{q_h}$ on each irreducible component of A. In particular, f has the pole of order $q_h \tilde{\delta}_k = \delta_k$ on each E_{j_k} $(0 \leq k \leq h-1)$. On the other hand, since $\Phi = 1 - f/g_h^{q_h}$ is non-constant on E_{i_h}, f has the pole of the same order $q_h \delta_h$ as $g_h^{q_h}$ on E_{i_h}. Hence, f has the pole of order δ_h on E_{j_h} by Lemma 1. Thus, $\{\delta_0,\ \delta_1,\ \cdots,\ \delta_h\}$ is the δ-sequence of the curve $C : f = 0$ with one place at infinity.

Q.E.D.

7 Refferences

[1] S.S.Abhyankar and T.T.Moh, Embeddings of the line in the plane, J. reine qngev. Math. 276(1975), 148-166.

[2] S.S.Abhyankar and T.T.Moh, On the semigroup of a meromorphic curve, Proc. Int. Symp. Algebraic Geometry, Kyoto (1977), 249-414.

[3] N.A'Campo and M.Oka, Geometry of plane curves via Tschirnhausen resolution tower, Osaka J. Math. Vol.33, No.4 (1996), 1003-1034.

[4] M.Furushima, Finite groups of polynomial automorphisms in the complex affine plane(I), Mem. Fac. Sci. Kyushu Univ., 36(1982), 85-105.

[5] M.Miyanishi, Minimization of the embeddings of the curves into the affine plane, J. Kyoto Univ. Vol.36, No.2 (1996), 311-329.

[6] J.A.Morrow, Compactifications of $\boldsymbol{C}^2$, Bull. Amer. Math. Soc. 78(1972), 813-816.

[7] Y.Nakazawa and M.Oka, Smooth plane curves with one place at infinity, to appear in J. Math. Soc. Japan, Vol.49, No.4 (1997).

[8] W.D.Neumann, Complex algebraic curves via their links at infinity, Invent. Math. 98(1989),445-489.
[9] C.P.Ramanujam, A topological characterization of the affine plane as an algebraic variety, Ann. Math. (2), 94(1971),69-88.
[10] A.Sathaye and J.Stenerson, On plane polynomial curves, Algebraic geometry and its applications, C.L.Bajaj, ed., Springer, (1994), 121-142.
[11] M.Suzuki, Propriété topologique des polynômes de deux variables complexes et automorphismes algébriques de l'espace $\boldsymbol{C}^2$, J. Math. Sci. Japan 26(1974),241-257.
[12] Zaidenberg and Lin, An irreducible simply connected algebraic curve in $\boldsymbol{C}^2$ is equivalent to a quasihomogeneous curve, Soviet Math. Dokl. Vol.28, No.1 (1983), 200-204.