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# CONSTRUCTION OF AFFINE PLANE CURVES WITH ONE PLACE AT INFINITY

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## 1. Introduction

Let $C$ be an irreducible algebraic curve in complex affine plane $\mathbf{C}^2$. We say that $C$ has *one place at infinity*, if the closure of $C$ intersects with the $\infty$-line in $\mathbf{P}^2$ at only one point $P$ and $C$ is locally irreducible at that point $P$.

The problem of finding the canonical models of curves with one place at infinity under the polynomial transformations of the coordinates of $\mathbf{C}^2$ has been studied by many mathematicians since Suzuki [17] and Abhyankar-Moh [2] proved independently that the canonical model of $C$ is a line when $C$ is non-singular and simply connected. Zaidenberg-Lin [19] proved that $C$ has the canonical model of type $y^q = x^p$, where $p$ and $q$ are coprime integers $> 1$, when $C$ is singular and simply connected. A'Campo-Oka [5] studied the case of genus $g \leq 3$ as an application of a resolution tower of toric modifications. For the case $g \leq 4$ Neumann [12] studied from the viewpoint of the link at infinity, and Miyanishi [9] studied from the algebrico-geometric viewpoint. Nakazawa-Oka [11] gave the classifications of all the canonical models for the case $g \leq 7$ using the result of A'Campo-Oka, and gave the classifications for the case $g \leq 16$ without proof. Jaworski [8] studied normal forms of irreducible germs of functions of two variables with given Puiseux pairs. Oka [14, 15] gave the normal form of plane curves which are locally irreducible at the origin and with a given sequence of weight vectors corresponding to the Tschirnhausen-good resolution tower, and showed that the moduli space of such curves is of the form $(\mathbf{C}^*)^a \times \mathbf{C}^b$. Furthermore, Oka translated this result to the case of affine curves with one place at infinity.

Also, Abhyankar-Moh [1, 3, 4] investigated properties of $\delta$-*sequences* which are sequences of pole orders of *approximate roots* of $C$. This result is called Abhyankar-Moh's semigroup theorem. Sathaye-Stenerson [16] proved that if a sequence $S$ of natural numbers satisfies Abhyankar-Moh's condition then there exists a curve with one place at infinity of the $\delta$-sequence $S$. Suzuki [18] made it clear the relationship between the $\delta$-sequence and the dual graph of the minimal resolution of the singularity of the curve $C$ at infinity, and gave an algebrico-geometric proof of semigroup theorem and its inverse theorem due to Sathaye-Stenerson.

In this paper, we develop Suzuki's result and give an algebrico-geometric proof of Oka's result (Theorem 7 and Corollary 1). We shall also give an algorithm to compute the normal form and the moduli space of the curve with one place at infinity from a given $\delta$-sequence[1].

Our construction method of normal forms is different from [8, 14, 15] in the following respects. First, this method uses $\delta$-sequences generating semigroups of affine plane curves with one place at infinity. Second, this method directly generates defining polynomials at the origin of curves with one place at infinity.

## 2. Preparations

In this section, we introduce some definitions and facts which is needed to describe our theorem.

Let $C$ be a curve with one place at infinity defined by a polynomial equation $f(x, y) = 0$ in the complex affine plane $\mathbf{C}^2$. Assume that $\deg_x f = m$, $\deg_y f = n$ and $d = \gcd(m, n)$. By the consideration of the Newton boundary, we can get

$$f(x, y) = (ux^p + vy^q)^d + \sum_{q\alpha+p\beta<pqd} c_{\alpha\beta}x^\alpha y^\beta,$$

where $u, v \in \mathbf{C}^*$, $m = pd$ and $n = qd$. By a finitely many times of the coordinate transformations of the form

$$\begin{cases} x_1 = x \\ y_1 = y + cx^p \end{cases}$$

and the exchange of the coordinates $x$ and $y$, we can reduce the polynomial $f$ into one of the following two types:

(A) $m = 1,\ n = 0$

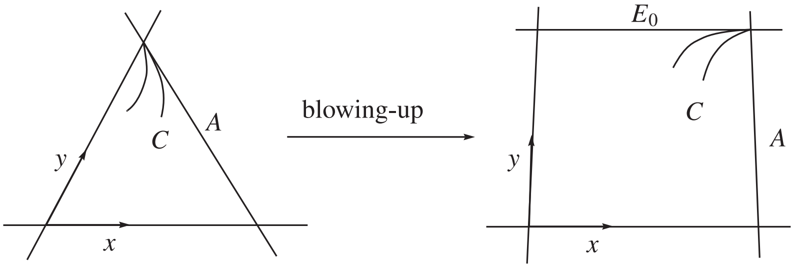
(B) $m = pd,\ n = qd,\ \gcd(p, q) = 1,\ p > q > 1.$

A curve of type (A) is a line. We call the curve of type (B) *non-linearlizable*. In this paper, we shall consider only the curves of the type (B) from now on. The closure $\overline{C}$ of $C$ in the projective plane $\mathbf{P}^2$ passes through the intersection point $O$ of the $\infty$-line $A$ and the line $x = 0$ by the assumption $p > q$.

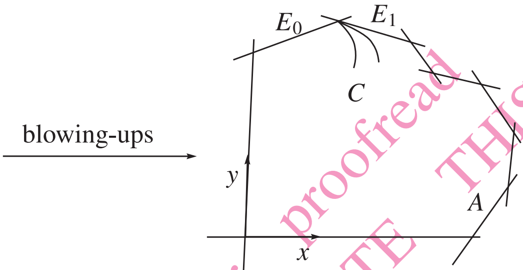
Let us denote by $E_0$ the $(-1)$-curve appeared by the blowing-up of the point $O$, and continue to denote the proper transform of $A$ by the same character $A$. Let $a$ be the natural number satisfying $aq < p < (a+1)q$. If $a = 1$, then the proper transform of $\overline{C}$ is tangent to $A$, or else is tangent to $E_0$.



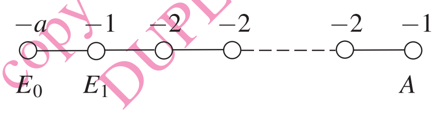
[1]The computer calculation by our algorithm verified the result of Nakazawa-Oka [11].



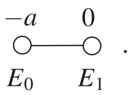

In case $a > 1$, after further $a - 1$ times of the blowing-ups of the point at infinity of the curve $C$, the proper transform of $\overline{C}$ is tangent to the $(-1)$-curve $E_1$ obtained by the last blowing-up. (In case $a = 1$, we set $E_1 = A$.)




Thus we get a compactification of $\mathbf{C}^2$ with the boundary curve of which the dual graph is of the following form:




By $a - 1$ times of the blowing-downs of the $(-1)$-curve on the right hand side from $A$ of the above dual graph, we get the following dual graph:

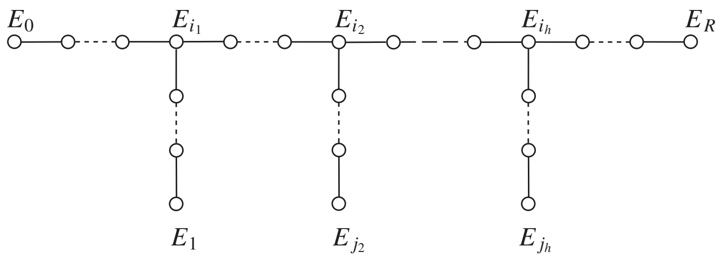



Let $(M_1,\ E_0 \cup E_1)$ be the compactification of $\mathbf{C}^2$ thus obtained.

The intersection point of $E_0$ and $E_1$ is the indetermination point of $f$. Now, we blow up from the surface $M_1$ the indetermination points of $f$ successively, until the indetermination points of $f$ disappear. Let $M_f$ be the surface thus obtained. We denote the proper transform in $M_f$ of $E_0$ (resp. $E_1$) by the same character $E_0$ (resp. $E_1$). Let $E_i\,(2 \leq i \leq R)$ be the proper transform in $M_f$ of the $(-1)$-curve obtained by the $(i-1)$-th blowing-up. Furthermore, we set $E_f = E_0 \cup E_1 \cup \cdots \cup E_R$.

The following theorem about the compactification $(M_f, E_f)$ of $\mathbf{C}^2$ is very important for the classification problem of the curves with one place at infinity.

**Theorem 1** ([18]). (i) *The dual graph* $\Gamma(E_f)$ *of* $E_f$ *has the following form*:



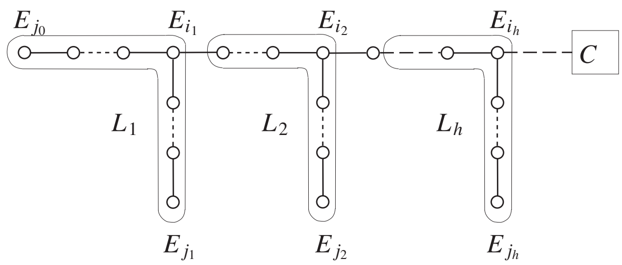

(ii) $f$ *is non-constant only on* $E_R$ *and has the pole on* $E_f - E_R$.
(iii) *The degree of* $f$ *on* $E_R$ *is* 1.
(iv) $E_R$ *is the unique* $(-1)$*-curve in* $E_f$.

**Note.** There is a small gap in the proof of (i) described in [18]. Let $Z$ (resp. $P$, $S$) be the union of the components of $E_f$ on which $f = 0$ (resp. $f = \infty$, $f$ =non-constant). Let $T$ be the union of the other components of $E_f$. From the proof of (i) described in [18], we know that $Z$ and $P$ are both connected and $S = E_R$. Here, since $f$ is non-zero constant on $T$, $T$ does not intersect $Z$ and $P$. If $T \neq \emptyset$, then $T$ intersects only $S$. But since $S(= E_R)$ is the last $(-1)$-curve on $M_f$, the relations of intersection among $Z$, $P$, $S$ and $T$ is one of the following two types:

(I) $P$—$S$—$Z$ (II) $P$—$S$—$T$.

If $Z \neq \emptyset$, then we get the contradiction as it is described in [18]. The similar argument applies to the case of $T \neq \emptyset$. Thus we get $Z = \emptyset$ and $T = \emptyset$. As a consequence, $\Gamma(E_f)$ has the above form.

In $\Gamma(E_f)$, let $i_1, i_2, \ldots, i_h$ (resp. $j_0, j_1, \ldots, j_h$) be the indices of the branch vertices (resp. the terminal vertices) from the left hand side, where $j_0 = 0$ and $j_1 = 1$. Let $M_C$ be the surface obtained by the blowing-down of $E_R, E_{R-1}, \ldots, E_{i_h+1}$ from $M_f$. For $i\,(0 \leq i \leq i_h)$, we shall continue to denote by $E_i$ the proper transform of $E_i$ in $M_C$. Further, we set $E_C = E_0 \cup E_1 \cup \cdots \cup E_{i_h}$. We shall call the pair $(M_C, E_C)$ the compactification of $\mathbf{C}^2$ obtained by the *minimal resolution* of the singularity of $C$ at infinity. We set $L_k = \bigcup_{i_{k-1} < i \leq i_k} E_i$ for each $k\,(1 \leq k \leq h)$ like the following figure, where $i_0 = -1$.

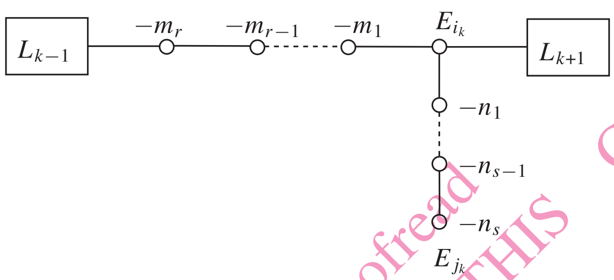


DEFINITION 1 ($\delta$-sequence). Let $\delta_k\,(0 \le k \le h)$ be the order of the pole of $f$ on $E_{j_k}$. We shall call the sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$ the *$\delta$-sequence* of $C$ (or of $f$).

We have the following fact since $\deg_x f = m$ and $\deg_y f = n$.

**Fact 1.** $\delta_0 = n,\ \delta_1 = m$.

DEFINITION 2 ($(p, q)$-sequence). Now, we assume that the weights of $L_k$ is of the following form:




We define the natural numbers $p_k,\ a_k,\ q_k,\ b_k$ satisfying

$$(p_k, a_k) = 1,\ (q_k, b_k) = 1,\ 0 < a_k < p_k,\ 0 < b_k < q_k,$$

$$\frac{p_k}{a_k} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}} \quad \text{and} \quad \frac{q_k}{b_k} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}}.$$

We shall call the sequence $\{(p_1, q_1), (p_2, q_2), \dots, (p_h, q_h)\}$ the *$(p, q)$-sequence* of $C$ (or of $f$).

We shall assume that $f(x, y)$ is monic in $y$. We define approximate roots by Abhyankar's definition.

DEFINITION 3 (approximate roots). Let $f(x, y)$ be the defining polynomial, monic in $y$, of a curve with one place at infinity. Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ be the $\delta$-sequence of $f$. We set $n = \deg_y f$, $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ and $n_k = n/d_k\ (1 \le k \le h+1)$. Then, for each $k\,(1 \le k \le h+1)$, a pair of polynomials $(g_k(x, y), \psi_k(x, y))$ satisfying the following conditions is uniquely determined:

(i) $g_k$ is monic in $y$ and $\deg_y g_k = n_k$,

(ii) $\deg_y \psi_k < n - n_k$,

(iii) $f = g_k^{d_k} + \psi_k$.

We call this $g_k$ the *$k$-th approximate root* of $f$.

We can easily get the following fact from the definition of approximate roots.

**Fact 2.** *We have*

$$g_1 = y + \sum_{j=0}^{\lfloor p/q \rfloor} c_k x^k, \quad g_{h+1} = f$$

*where* $c_k \in \mathbf{C}$, $p = \deg_x f/d$, $q = \deg_y f/d$, $d = \gcd\left\{\deg_x f, \deg_y f\right\}$ *and* $\lfloor p/q \rfloor$ *is the maximal integer* $l$ *such that* $l \leq p/q$.

DEFINITION 4 ($g$-sequence). The sequence of polynomials $g_0 := x,\ g_1, \ldots, g_{h+1}$ is called the *$g$-sequence* of $f$.

Here, we denote by $C_k$ the curve defined by $g_k(x, y) = 0$ in $\mathbf{C}^2$. The following theorem about $C_k$ plays a vital role in the main theorem.

**Theorem 2.** *For each* $k\,(0 \leq k \leq h)$, $C_k$ *is also with one place at infinity. Further, its closure* $\overline{C}_k$ *in* $M_C$ *intersects transversely* $E_{j_k}$, *and does not intersect other irreducible components of* $E_C$.

Suzuki [18] gave the algebrico-geometric proof of this theorem. We get the following theorem as a corollary of the above theorem.

**Theorem 3.** *For each* $k\,(0 \leq k \leq h)$, $g_k$ *has the pole of order* $\delta_k$ *on* $E_{i_h}$.

The following lemma about approximate roots will be used in Theorem 6.

**Lemma 1.** *Let* $f$ *be the defining polynomial, monic in* $y$, *of a curve with one place at infinity. Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ *be the* $\delta$-*sequence of* $f$, *and* $g_0, g_1, \ldots, g_h, g_{h+1}$ *be the* $g$-*sequence of* $f$. *Then,* $g_k\,(0 \leq k \leq h-1)$ *is also the* $k$-*th approximate root of* $g_j$ *for any* $j$ *with* $k < j < h+1$.

Proof. For example, see Proposition 2.2 in [5]. □



## 3. Intersection matrix and successive blow-up

Let $M$ be a non-singular projective algebraic surface over complex number field, and $E$ be an algebraic curve on $M$. We shall assume that $E_1, E_2, \ldots, E_s$ are irreducible components of $E$, and denote by $I_E$ the intersection matrix $((E_i \cdot E_j))_{i,j=1,\ldots,s}$ of $E$. The following lemma about the intersection matrix is well-known by Mumford.

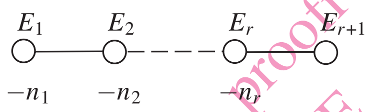
**Lemma 2.** *$E$ is an exceptional set if and only if $I_E$ is negative definite.*

Let $E_1'$ be the $(-1)$-curve appeared by blowing-up at a point $P_0$ on a surface $M$, and let $P_1$ be a point on $E_1'$. For $i\,(\geq 1)$, let $E_{i+1}'$ be the $(-1)$-curve appeared by blowing-up at a point $P_i$, and let $P_{i+1}$ be the point on $E_{i+1}'$. We get $\{P_i\}_{i=0,\dots,r}$ and $\{E_i'\}_{i=1,\dots,r}$ by the above finite operations. In this paper we call this finite sequence of blowing-ups a *successive blow-up from* $P_0$. Let $M'$ be the surface obtained by a successive blow-up from $P_0$. For $i\,(1 \leq i \leq r)$, we shall continue to denote by $E_i'$ the proper transform of $E_i'$ in $M'$. Further, we set $E' = \bigcup_{i=1}^r E_i'$ and $\Delta_{E'} = \det(-I_{E'})$. We have the following fact since $\Delta_{E'}$ is invariant under the successive blow-up.

**Fact 3.** $\Delta_{E'} = 1$.

The following lemma is Lemma 1 in [18]. Here, we describe it because it is used many times in the next section.

**Lemma 3.** *Let $E_1, E_2, \dots, E_r, E_{r+1}$ be the irreducible components of $E$ and assume that the dual graph $\Gamma(E)$ is of the following linear type:*



$(n_i \geq 2)$.

*Assume further that there exists a holomorphic function $f$ on a neighborhood $U$ of $\bigcup_{i=1}^r E_i$ such that the zero divisor $(f)$ of $f$ on $U$ is written in the following form:*

$$\sum_{i=1}^{r} m_i E_i + m_{r+1} E_{r+1} \cap U\,.$$

*Let $(p_i, p_{i+1})$ be the coprime integers defined by the following continued fraction:*

$$\frac{p_{i+1}}{p_i} = n_i - \cfrac{1}{n_{i-1} - \ddots - \cfrac{1}{n_1}} \quad (1 \leq i \leq r).$$

*Then, $m_i = m_1 p_i$ $(1 \leq i \leq r+1)$.*

Now, consider a pair of natural numbers $(p, q)$ with $\gcd(p, q) = 1$, $p > q > 0$. We can easily show that there exists a unique pair of natural numbers $(a, b)$ with $pq - aq - bp = 1$, $0 < a < p$, $0 < b < q$.

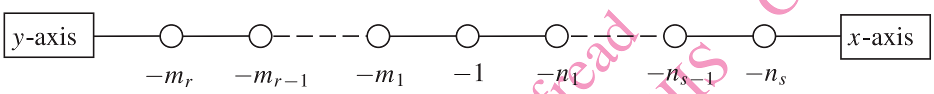
We consider the following continued fractions for the above mentioned $p$, $q$, $a$, $b$:

$$\frac{p}{a} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}}, \quad \frac{q}{b} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}},$$

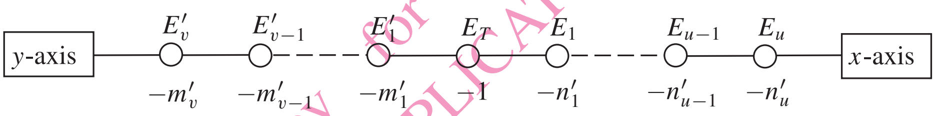
where $m_i \geq 2$ and $n_j \geq 2$.

Let $(x, y)$ be the local coordinate for the neighborhood of a point $P$ on $M$ which has $P$ as the origin. Then,

**Lemma 4.** *we can construct a exceptional curve with the following weights by a successive blow-up from $P$.*




Proof. We consider the curve $C$ defined by $x^p + y^q = 0$. The resolution graph at origin of $C$ is as follows:




Let $I_E$ be the intersection matrix of the exceptional curve $E$ corresponding to the above dual graph. Here, we set

$$\frac{p'}{a'} = m'_1 - \cfrac{1}{m'_2 - \ddots - \cfrac{1}{m'_u}}, \quad \frac{q'}{b'} = n'_1 - \cfrac{1}{n'_2 - \ddots - \cfrac{1}{n'_v}}.$$

We get $\det(-I_E) = p'q' - a'q' - b'p'$. On the other hand, $E$ is the exceptional curve obtained by a successive blow-up from origin. Therefore, we get $\det(-I_E) = 1$ by Fact 3. Thus $p'q' - a'q' - b'p' = 1$.

As the above dual graph, let $E_i$ $(1 \leq i \leq u)$, $E_T$, $E'_j$ $(1 \leq j \leq v)$ be the irreducible components of $E$. We denote by $\mu_i$ $(1 \leq i \leq u)$ the zero order of the function $x$ on $E_i$ and by $\mu_T$ the zero order of the function $x$ on $E_T$. Also, we denote by $\nu_j$ $(1 \leq j \leq v)$ the zero order of the function $y$ on $E'_j$ and by $\nu_T$ the zero order of the function $y$ on $E_T$. Since $q = \mu_T$ and $\mu_u = 1$, we get $q' = \mu_T/\mu_u = q$ by Lemma 3. As the same way, we get $p = p'$. Thus $pq - a'q - b'p = 1$. Further, it must be $a = a'$, $b = b'$, since $0 < a' < p$ and $0 < b' < q$. Therefore, we get $v = r$, $m'_i = m_i$ $(1 \leq i \leq r)$, $u = s$,

$n'_j = n_j\,(1 \le j \le s)$ by the uniqueness of the expansion into continued fraction. As a result, the assertion was proved. □

## 4. Construction of a curve with one place at infinity

We set $\mathbf{N} = \{n \in \mathbf{Z} \mid n \ge 0\}$ and $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$. The following theorem about $\delta$-sequence and $(p, q)$-sequence is called Abhyankar-Moh's Semigroup Theorem.

**Theorem 4** (Abhyankar-Moh). *Let $C$ be a non-linearizable affine plane curve with one place at infinity. Let $\{\delta_0, \delta_1, \dots, \delta_h\}$ be the $\delta$-sequence of $C$ and $\{(p_1, q_1), \dots, (p_h, q_h)\}$ be the $(p, q)$-sequence of $C$. We set $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ $(1 \le k \le h+1)$. We have then,*

(i) $q_k = d_k/d_{k+1},\ d_{h+1} = 1\,(1 \le k \le h)$,

(ii) $d_{k+1}p_k = \begin{cases} \delta_1 & (k=1) \\ q_{k-1}\delta_{k-1} - \delta_k & (2 \le k \le h) \end{cases}$,

(iii) $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \cdots + \mathbf{N}\delta_{k-1}\ (1 \le k \le h)$.

The following theorem gives the converse of the above theorem.

**Theorem 5** (Sathaye-Stenerson [16]). *Let $\{\delta_0, \delta_1, \dots, \delta_h\}\,(h \ge 1)$ be the sequence of $h+1$ natural numbers. We set $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}\,(1 \le k \le h+1)$ and $q_k = d_k/d_{k+1}\,(1 \le k \le h)$. Furthermore, suppose that the following conditions are satisfied*:

(1) $\delta_0 < \delta_1$,

(2) $q_k \ge 2\,(1 \le k \le h)$,

(3) $d_{h+1} = 1$,

(4) $\delta_k < q_{k-1}\delta_{k-1}\,(2 \le k \le h)$,

(5) $q_k\delta_k \in \mathbf{N}\delta_0 + \mathbf{N}\delta_1 + \cdots + \mathbf{N}\delta_{k-1}\ (1 \le k \le h)$.

*Then, there exists a curve with one place at infinity of the $\delta$-sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$.*

Suzuki [18] gave an algebrico-geometric proof of the above two theorem by the consideration of the resolution graph at infinity.

DEFINITION 5 (Abhyankar-Moh's condition). We shall call the conditions (1)–(5) concerning $\{\delta_0, \delta_1, \dots, \delta_h\}$ in Theorem 5 *Abhyankar-Moh's condition*.

**Theorem 6.** *Let $\{\delta_0, \delta_1, \dots, \delta_h\}\,(h \ge 1)$ be the sequence of $h+1$ natural numbers satisfying Abhyankar-Moh's condition. Set $d_k = \gcd\{\delta_0, \delta_1, \dots, \delta_{k-1}\}$ $(1 \le k \le h+1)$ and $q_k = d_k/d_{k+1}\,(1 \le k \le h)$. Then,*

(i) *the defining polynomial $f$, monic in $y$, of a curve with one place at infinity of* *the $\delta$-sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$ has the following form using the approximate roots $g_0, g_1, \dots, g_h$ of $f$:*

$$f = g_h^{q_h} + a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{h-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \cdots g_{h-1}^{\bar{\alpha}_{h-1}} + \sum_{(\alpha_0, \alpha_1, \dots, \alpha_h)\in\Lambda} c_{\alpha_0\alpha_1\cdots\alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h}$$

*where $a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{h-1}} \in \mathbf{C}^*$, $c_{\alpha_0\alpha_1\cdots\alpha_h} \in \mathbf{C}$, $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{h-1})$ is the sequence of $h$ non-negative integers satisfying*

$$\sum_{i=0}^{h-1} \bar{\alpha}_i \delta_i = q_h \delta_h, \ \bar{\alpha}_i < q_i \ (0 < i < h)$$

*and*

$$\Lambda = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_h) \in \mathbf{N}^{h+1} \ \middle| \ \alpha_i < q_i \ (0 < i < h), \alpha_h < q_h - 1, \sum_{i=0}^{h} \alpha_i \delta_i < q_h \delta_h \right\}.$$

(ii) *Conversely, let $g_h$ be the defining polynomial, monic in $y$, of a curve with one place at infinity of the $\delta$-sequence $\{\delta_0/q_h, \delta_1/q_h, \dots, \delta_{h-1}/q_h\}$, and $g_0, g_1, \dots, g_{h-1}$ be the approximate roots of $g_h$. For any non-zero complex number $a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{h-1}}$ corresponding to the sequence of $h$ non-negative integers $(\bar{\alpha}_0, \bar{\alpha}_1, \dots, \bar{\alpha}_{h-1})$ satisfying*

$$\sum_{i=0}^{h-1} \bar{\alpha}_i \delta_i = q_h \delta_h, \quad \bar{\alpha}_i < q_i \ (0 < i < h)$$

*and any complex numbers $c_{\alpha_0\alpha_1\cdots\alpha_h}$ corresponding to the sequences of $h+1$ non-negative integers $(\alpha_0, \alpha_1, \dots, \alpha_h)$ satisfying*

$$\sum_{i=0}^{h} \alpha_i \delta_i < q_h \delta_h, \quad \alpha_i < q_i \ (0 < i < h), \quad \alpha_h < q_h - 1,$$

*we consider*

$$f = g_h^{q_h} + a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{h-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \cdots g_{h-1}^{\bar{\alpha}_{h-1}} + \sum_{(\alpha_0, \alpha_1, \dots, \alpha_h)\in\Lambda} c_{\alpha_0\alpha_1\cdots\alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h}$$

*where*

$$\Lambda = \left\{ (\alpha_0, \alpha_1, \dots, \alpha_h) \in \mathbf{N}^{h+1} \ \middle| \ \alpha_i < q_i \ (0 < i < h), \alpha_h < q_h - 1, \sum_{i=0}^{h} \alpha_i \delta_i < q_h \delta_h \right\}.$$

*Then, the curve defined by $f = 0$ is a curve with one place at infinity of the $\delta$-sequence $\{\delta_0, \delta_1, \dots, \delta_h\}$, and has the approximate roots $g_0, g_1, \dots, g_h$.*

Proof of Theorem 6. We shall prove (i). By the procedure described in the proof of Proposition 10 in [18], using the approximate roots $g_0, g_1, \ldots, g_h$ of $f$ and the set of $h+1$ non-negative integers $(\alpha_0, \alpha_1, \ldots, \alpha_h)$ with $\max\{\sum_{i=0}^{h} \alpha_i \delta_i\} = q_h \delta_h$, we can write $f$ as follows:

$$f = \sum_{\alpha_i < q_i (1 \le i \le h)} c_{\alpha_0 \alpha_1 \cdots \alpha_h} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h} + g_h^{q_h}, \quad c_{\alpha_0 \alpha_1 \cdots \alpha_h} \in \mathbf{C}.$$

Here, we suppose $f = g_h^{q_h} + g_h^{q_h - 1}$. We have $\deg_y g_h^{q_h-1} = n_h(q_h - 1) = \deg_y f - n_h = n - n_h$. But this is a contradiction, since $g_h$ is $h$-th approximate root of $f$. Thus we get $\alpha_h < q_h - 1$. By Theorem 4(iii) and the uniqueness of $\{\alpha_i\}_{i=0,\ldots,h}$ (e.g., Lemma 7 in [18]), we have $\{\bar{\alpha}_i\}_{i=0,\ldots,h-1}$ with $\sum_{i=0}^{h-1} \bar{\alpha}_i \delta_i = q_h \delta_h$. As a result, (i) was proved.

We shall prove (ii).

CASE $h = 1$. Set $\delta_0 = q$ and $\delta_1 = p$. We can write $f$ as follows:

$$f = y^q + ax^p + \sum_{q\alpha + p\beta < pq} c_{\alpha\beta} x^\alpha y^\beta, \quad a \in \mathbf{C}^*, \quad c_{\alpha\beta} \in \mathbf{C}.$$

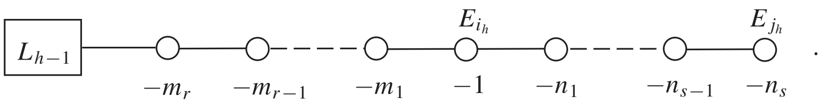
The curve defined by $f = 0$ has one place at infinity of the $\delta$-sequence $\{q, p\}$ by the consideration of Newton boundary.

CASE $h \ge 2$. Set $\delta_i / q_h = \tilde{\delta}_i$ $(0 \le i \le h-1)$. We denote by $C_k$ the curve defined by $g_k = 0$ for each $k$ with $0 \le k \le h$. Further, we shall denote by $(\tilde{M}, \tilde{E})$ the compactification of $\mathbf{C}^2$ obtained by the minimal resolution of $C_h$ at infinity. Let $\tilde{C}_k$ be the proper transform of $C_k$ on $\tilde{M}$ and $\tilde{E}_i$ be the irreducible components of $\tilde{E}$. (The way of numbering about indices is same as Section 2.) By Theorem 2, $\tilde{C}_k$ has one place at infinity and intersects transversely $\tilde{E}_{j_k}$ $(0 \le k \le h-1)$.

Let $Q$ be the intersection point of $\tilde{C}_h$ and $\tilde{E}_{i_{h-1}}$. Set $p_h = q_{h-1}\delta_{h-1} - \delta_h$. ($p_h > 0$ since Abhyankar-Moh's condition (4).) We have $\gcd(p_h, q_h) = 1$ from $\gcd(q_h, \delta_h) = d_{h+1} = 1$ and get a unique pair of natural numbers $(a_h, b_h)$ with $p_h q_h - a_h q_h - b_h p_h = 1$, $0 < a_h < p_h$, $0 < b_h < q_h$. We define $\{m_i\}_{i=1,\ldots,r}$, $\{n_j\}_{j=1,\ldots,s}$ using the following expansion into continued fractions by $p_h$, $a_h$, $q_h$, $b_h$:

$$\frac{p_h}{a_h} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \ddots - \cfrac{1}{m_r}}}, \qquad \frac{q_h}{b_h} = n_1 - \cfrac{1}{n_2 - \cfrac{1}{n_3 - \ddots - \cfrac{1}{n_s}}}.$$

By Lemma 4, we can obtain the following branch $L_h$ such that $C_h$ intersects transversely $E_{j_h}$ using the successive blow-up from $Q$:




Let $M$ be the surface thus obtained, $E$ be the total transform of $\tilde{E}$ on $M$. We denote by $E_i$ (resp. $\overline{C}_k$) the proper transform of $\tilde{E}_i$ (resp. $\tilde{C}_k$).

Set $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \le k \le h+1)$ and $q_k = d_k/d_{k+1}$ $(1 \le k \le h)$. By Theorem 3, $g_k$ has the pole of order $\tilde{\delta}_k$ on $E_{i_{h-1}}$ for each $k$ $(0 \le k \le h-1)$. Thus $g_k$ has the pole of order $\tilde{\delta}_k$ on $E_{j_h}$ and of order $q_h\tilde{\delta}_k (= \delta_k)$ on $E_{i_h}$. On the other hand, $g_h$ has the pole of order $\delta_h$ on $E_{i_h}$. In fact, we can write $g_h$ on a neighborhood of $Q$ as follows:

$$g_h = \frac{v}{u^{q_{h-1}\tilde{\delta}_{h-1}}} \times \text{(non-const)}.$$

Hence $g_h$ has the pole of order $q_h(q_{h-1}\tilde{\delta}_{h-1}) - p_h$ on $E_{i_h}$. This value is equal to $\delta_h$ by the assumption of $p_h$.

Now, we consider the curve $C$ defined by $f = 0$. Set $\phi = f - g_h^{q_h}$ and $\Phi = \phi/g_h^{q_h}$. Since the both of $g_h^{q_h}$ and $\phi$ has the pole of order $q_h\delta_h$ on $E_{i_h}$, $\Phi$ is non-constant or constant$(\neq 0)$ on $E_{i_h}$.

Let $A$ (resp. $B$) be the closure of the connected component of $E - E_{i_h}$ which contains $E_0$ (resp. $E_{j_h}$). Let $P_{g_h}$ be the pole divisor of $g_h$ on $M$, and $D$ be its restriction to $A$. Here, let $F_1$ be the irreducible component of $A$ intersecting $E_{i_h}$. Since $g_h$ has the pole of order $\delta_h$ on $E_{i_h}$, we have $(D \cdot F_1) < 0$. Also, since $(D \cdot E_i) = 0$ for any $E_i$ with $E_i \neq F_1$, using Proposition 2 in [6], the intersection matrix of $A$ is negative definite. Thus it follows that $A$ is exceptional set. $\Phi$ is holomorphic on $A$ since $A \cap \overline{C}_h = \emptyset$. On the other hand,

$$\begin{aligned}
&\deg_y g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h} \\
&\quad = \sum_{i=0}^{h} \alpha_i n_i = \sum_{i=1}^{h} \alpha_i n_i \\
&\quad = \alpha_1 + \alpha_2 q_1 + \alpha_3 q_2 q_1 + \cdots + \alpha_h q_{h-1} \cdots q_1 \\
&\quad < (q_1 - 1) + (q_2 - 1)q_1 + (q_3 - 1)q_2 q_1 + \cdots + (q_h - 1)q_{h-1} \cdots q_1 \\
&\quad = q_h q_{h-1} \cdots q_1 - 1 \\
&\quad < q_h q_{h-1} \cdots q_1 = q_h n_h = \deg_y g_h^{q_h}.
\end{aligned}$$

Therefore, we get $\deg_y \phi < \deg_y g_h^{q_h}$. Hence, $\Phi = 0$ on $E_0$. Further, $\Phi = 0$ on $A$, since $A$ is compact. As a result, it must be that $\Phi$ is non-constant on $E_{i_h}$.

Let $P_\Phi$ be the pole divisor of $\Phi$ on $M$. We denote by $B_1, B_2, \ldots, B_s$ the irreducible components of $B$ in order from the component intersecting $E_{i_h}$. Since $\Phi$ has the pole on $B_1$ and $\overline{C}_h$, the support of $P_\Phi$ is $B \cup \overline{C}_h$ and we can write $P_\Phi =$

$q_h\overline{C}_h + \sum_{i=1}^{s} \mu_i B_i$ ($\mu_i > 0$). By

$$n_s - \cfrac{1}{n_{s-1} - \ddots - \cfrac{1}{n_1}} = \frac{q_h}{b'}$$

and Lemma 3, we get $\mu_1 q_h = q_h$, where $\mu_1$ is the pole order of $\Phi$ on $B_1$. Hence, $\mu_1 = 1$. This implies that $\Phi$ is a rational function of degree 1 on $E_{i_h}$. Therefore, the curve defined by $\Phi = -1$ intersects transversely $E_{i_h}$ at only one point. Since the curve $\Phi = -1$ coincides with $\overline{C}$, we get

$$(\overline{C} \cdot E_i) = \begin{cases} 1 & (i = i_h) \\ 0 & (i \neq i_h) \end{cases}.$$

As a result, $C$ has one place at infinity.

We have $f = g_h^{q_h}$ on $A$, since $\Phi = 0$ on $A$. Hence, $f$ has the pole of the same order as $g_h^{q_h}$ on each irreducible component of $A$. In particular, $f$ has the pole of order $q_h\tilde{\delta}_k = \delta_k$ on each $E_{i_k}$ ($0 \leq k \leq h-1$). Since $\Phi$ is non-constant on $E_{i_h}$, $f$ has the pole of the same order as $g_h^{q_h}$ on $E_{i_h}$. Since the value of its pole order is $q_h\delta_h$, using Lemma 3, it follows that $f$ has the pole of order $\delta_h$ on $E_{j_h}$. Consequently, $\{\delta_0, \delta_1, \ldots, \delta_h\}$ is the $\delta$-sequence of $f$.

Finally, we show that $g_0, g_1, \ldots, g_h$ are the approximate roots of $f$. By

$$\begin{aligned} &\deg_y g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_h^{\alpha_h} \\ &\quad = n_0\alpha_0 + n_1\alpha_1 + \cdots + n_h\alpha_h \\ &\quad \leq n_1(q_1 - 1) + n_2(q_2 - 1) + \cdots + n_{h-1}(q_{h-1} - 1) + n_h(q_h - 2) \\ &\quad = -n_1 + n_h q_h - n_h < n - n_h, \end{aligned}$$

$g_h$ is $h$-th approximate root of $f$. Therefore, by Lemma 1, $g_0, g_1, \ldots, g_h$ are the approximate roots of $f$. □

The following theorem is the main theorem in this paper, and is obtained by using Theorem 6 inductively.

**Theorem 7.** *Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ $(h \geq 1)$ *be a sequence of natural numbers satisfying Abhyankar-Moh's condition (see* Definition 5*). Set* $d_k = \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \leq k \leq h+1)$ *and* $q_k = d_k/d_{k+1}$ $(1 \leq k \leq h)$.

(1) *We define* $g_k\,(0 \le k \le h+1)$ *as follows*:

$$\begin{cases} g_0 = x, \\ g_1 = y + \displaystyle\sum_{j=0}^{\lfloor p/q \rfloor} c_j x^j, \quad c_j \in \mathbf{C},\ p = \frac{\delta_1}{d_2},\ q = \frac{\delta_0}{d_2}, \\ g_{i+1} = g_i^{q_i} + a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}} g_0^{\bar{\alpha}_0} g_1^{\bar{\alpha}_1} \cdots g_{i-1}^{\bar{\alpha}_{i-1}} \\ \qquad + \displaystyle\sum_{(\alpha_0,\alpha_1,\ldots,\alpha_i)\in\Lambda_i} c_{\alpha_0\alpha_1\cdots\alpha_i} g_0^{\alpha_0} g_1^{\alpha_1} \cdots g_i^{\alpha_i}, \\ \qquad a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}} \in \mathbf{C}^*,\ c_{\alpha_0\alpha_1\cdots\alpha_i} \in \mathbf{C} \quad (1 \le i \le h), \end{cases}$$

*where* $(\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1})$ *is the sequence of* $i$ *non-negative integers satisfying*

$$\sum_{j=0}^{i-1} \bar{\alpha}_j \delta_j = q_i \delta_i,\ \bar{\alpha}_j < q_j\ (0 < j < i)$$

*and*

$$\Lambda_i = \left\{ (\alpha_0, \alpha_1, \ldots, \alpha_i) \in \mathbf{N}^{i+1} \;\middle|\; \alpha_j < q_j\ (0 < j < i),\ \alpha_i < q_i - 1,\ \sum_{j=0}^{i} \alpha_j \delta_j < q_i \delta_i \right\}.$$

*Then*, $g_0, g_1, \ldots, g_h$ *are approximate roots of* $f\,(= g_{h+1})$, *and* $f$ *is the defining polynomial, monic in* $y$, *of a curve with one place at infinity of the* $\delta$*-sequence* $\{\delta_0, \delta_1, \ldots, \delta_h\}$.

(2) *The defining polynomial* $f$, *monic in* $y$, *of a curve with one place at infinity of the* $\delta$*-sequence* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ *is obtained by the procedure of* (1), *and the values of parameters* $\{a_{\bar{\alpha}_0\bar{\alpha}_1\cdots\bar{\alpha}_{i-1}}\}_{1\le i\le h}$ *and* $\{c_{\alpha_0\alpha_1\cdots\alpha_i}\}_{0\le i\le h}$ *are uniquely determined for* $f$.



The above theorem gives normal forms of defining polynomials of curves with one place at infinity and the method of construction of their defining polynomials.

**Corollary 1.** *Let* $\{\delta_0, \delta_1, \ldots, \delta_h\}\,(h \ge 1)$ *be a sequence of natural numbers satisfying Abhyankar-Moh's condition. The moduli space of the curve* $C$ *with one place at infinity of the* $\delta$*-sequence* $\{\delta_0, \delta_1, \ldots, \delta_h\}$ *is isomorphic to*

$$(\mathbf{C}^*)^h \times \mathbf{C}^b,$$

*where* $b$ *is the total number of parameters* $\{c_{\alpha_0\alpha_1\cdots\alpha_i}\}_{0\le i\le h}$ *appeared in the defining polynomial, monic in* $y$, *of* $C$ *obtained in* Theorem 7.

Proof. We consider the defining polynomial $f$, monic in $y$, of the curve $C$ with one place at infinity of the $\delta$-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$. We denote by $a$ the number of non-zero parameters in $f$ and by $b$ the number of others. By Theorem 7, the moduli space of $C$ is $(\mathbf{C}^*)^a \times \mathbf{C}^b$. $f$ has $h+2$ polynomials $g_0, g_1, \ldots, g_{h+1}$. Here, both of $g_0$ and $g_1$ do not have non-zero parameter. Also, $g_{i+1}$ $(1 \le i \le h)$ has exactly one non-zero parameter because the sequence of $i+1$ non-negative integers $(\alpha_0, \alpha_1, \ldots, \alpha_i)$ with $\sum_{j=0}^{i} \alpha_j \tilde{\delta}_j = q_i \tilde{\delta}_i$ is determined uniquely. As a result, we get $a = h$. □

By the above results, we can easily get an algorithm generating the defining polynomial and computing the moduli space from a $\delta$-sequence. We will introduce them in the next section.

## 5. Algorithms

Using Theorem 7, the following algorithm generating the defining polynomial of the curve with one place at infinity from a $\delta$-sequence is obtained.

**Algorithm 1**: generating polynomial
**Input**: $\delta$-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$
**Output**: the defining polynomial $f(x, y)$ of the curve with one place at infinity of the $\delta$-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$

$D \leftarrow [\delta_h, \delta_{h-1}, \ldots, \delta_0]$
$d_k \leftarrow \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}$ $(1 \le k \le h+1)$
$Q \leftarrow [q_h, \ldots, q_1]$ where $q_k = d_k/d_{k+1}$ $(1 \le k \le h)$
$DL \leftarrow cons(D, [\,])$
$QL \leftarrow cons(Q, [\,])$
$m \leftarrow h + 1$
**while** $m \ne 2$ **do**
    $T \leftarrow reverse(cdr(D))$
    $D \leftarrow [\,]$
    **while** $T \ne [\,]$ **do**
        $D \leftarrow cons(car(T)/car(Q), D)$
        $T \leftarrow cdr(T)$
    **end**
    $DL \leftarrow cons(D, DL)$
    $Q \leftarrow cdr(Q)$
    $QL \leftarrow cons(Q, QL)$
    $m \leftarrow length(D)$
**end**
$AL \leftarrow [x]$
$D \leftarrow car(DL)$
$l \leftarrow \lfloor car(D)/car(cdr(D)) \rfloor$
$g_1 \leftarrow y + \sum_{j=0}^{l} c_j x^j$

$AL \leftarrow cons(g_1, AL)$
**while** $DL \neq [\ ]$ **do**
$D \leftarrow car(DL)$
$Q \leftarrow car(QL)$
$q_0 \leftarrow \lfloor car(Q) \times car(D)/car(reverse(D)) \rfloor + 1$
$L \leftarrow append(Q, [q_0])$
$k \leftarrow length(D) - 1$, i.e., $D = [\bar{\delta}_k, \ldots, \bar{\delta}_0]$, $L = [q_k, \ldots, q_0]$.
$(\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{k-1}) \leftarrow$ the sequence of non-negative integers with $\sum_{i=0}^{k-1} \bar{\alpha}_i \bar{\delta}_i = \bar{\delta}_k q_k$, $\bar{\alpha}_i < q_i$ $(0 \leq i \leq k-1)$, $\bar{\delta}_i \in D$ and $q_i \in L$
$\{(\alpha_0, \alpha_1, \ldots, \alpha_k)\} \leftarrow$ the set of sequences of non-negative integers with $\sum_{i=0}^{k} \alpha_i \bar{\delta}_i < \bar{\delta}_k q_k$, $\alpha_i < q_i$ $(0 \leq i < k)$, $\alpha_k < q_k - 1$, $\bar{\delta}_i \in D$ and $q_i \in L$
$g_{k+1} \leftarrow g_k^{q_k} + a_{\bar{\alpha}_0, \bar{\alpha}_1, \ldots, \bar{\alpha}_{k-1}} \prod_{i=0}^{k-1} g_i^{\bar{\alpha}_i} + \sum c_{\alpha_0, \alpha_1, \ldots, \alpha_k} \prod_{i=0}^{k} g_i^{\alpha_i}$
$AL \leftarrow cons(g_{k+1}, AL)$
$DL \leftarrow cdr(DL)$
$QL \leftarrow cdr(QL)$
**end**
**return** $car(AL)$

SUPPLEMENTATION:
- $[\ldots]$ := A list. (This is a data structure with ordered elements.)
- $\lfloor p \rfloor$ := The maximal integer $n$ such that $n \leq p$.
- $car(L)$ := The first element of a given non-null list $L$.
- $cdr(L)$ := The list obtained by removing the first element of a given non-null list $L$.
- $cons(A, L)$ := The list obtained by adding an element $A$ to the top of a given list $L$.
- $reverse(L)$ := The reversed list of a given list $L$.
- $append(L_1, L_2)$ := The list obtained by adding all elements in a list $L_2$ according to the order as it is to the last element in a list $L_1$.
- $length(L)$ := The number of elements of a given list $L$.
- $a_{*,*,\ldots,*}$ is a parameter in $\mathbf{C}^*$.
- $c_{*,*,\ldots,*}$ is a parameter in $\mathbf{C}$.

The moduli space of $f$ is obtained by counting the numbers of $\{a_{*,*,\ldots,*}\}$ and $\{c_{*,*,\ldots,*}\}$ in $f$ which the above algorithm outputted. But we can compute the moduli space from a $\delta$-sequence without generating the defining polynomial. The following algorithm directly compute the moduli space from a $\delta$-sequence.

**Algorithm 2**: computation of moduli space
**Input**: $\delta$-sequence $\{\delta_0, \delta_1, \ldots, \delta_h\}$
**Output**: $[M, N]$ (This means the moduli space $(\mathbf{C}^*)^M \times \mathbf{C}^N$.)

$D \leftarrow [\delta_h, \delta_{h-1}, \ldots, \delta_0]$

$d_k \leftarrow \gcd\{\delta_0, \delta_1, \ldots, \delta_{k-1}\}\,(1 \le k \le h+1)$
$Q \leftarrow [q_h, \ldots, q_1]$ where $q_k = d_k/d_{k+1}\,(1 \le k \le h)$
$Q \leftarrow cons(1, Q)$
$M \leftarrow h$
$N \leftarrow 0$
**while true do**
  $k \leftarrow length(D) - 1$, i.e., $D = [\bar{\delta}_k, \ldots, \bar{\delta}_0]$
  $D \leftarrow [\bar{\delta}_k/car(Q), \bar{\delta}_{k-1}/car(Q), \ldots, \bar{\delta}_0/car(Q)]$
  $Q \leftarrow cdr(Q)$
  $q_0 \leftarrow \lfloor car(Q) \times car(D)/car(reverse(D)) \rfloor + 1$
  $L \leftarrow append(Q, [q_0])$, i.e., $L = [q_k, \ldots, q_0]$
  $n \leftarrow$ the number of $(\alpha_0, \alpha_1, \ldots, \alpha_k)$ with $\sum_{i=0}^{k} \alpha_i \bar{\delta}_i < car(Q) \times car(D)$,
    $\alpha_i < q_i\ (0 \le i \le k-1), \alpha_k < q_k - 1, \bar{\delta}_i \in D$ and $q_i \in L$
  $N \leftarrow N + n$
  **if** $length(D) = 2$ **then break**
  $D \leftarrow cdr(D)$
**end**
$N \leftarrow N + \lfloor p/q \rfloor + 1$
**return** $[M, N]$

## 6. Polynomial curve

**6.1. Abhyankar's question.** In this section, we will introduce Abhyankar's question.

DEFINITION 6 (planar semigroup). Let $\{\delta_0, \delta_1, \ldots, \delta_h\}\,(h \ge 1)$ be a sequence of natural numbers satisfying Abhyankar-Moh's condition. A semigroup generated by $\{\delta_0, \delta_1, \ldots, \delta_h\}$ is said to be *a planar semigroup*.

DEFINITION 7 (polynomial curve). Let $C$ be an algebraic curve defined by $f(x, y) = 0$, where $f(x, y)$ is an irreducible polynomial in $\mathbf{C}[x, y]$. We call $C$ *a polynomial curve*, if $C$ has a parametrisation $x = x(t)$, $y = y(t)$, where $x(t)$ and $y(t)$ are polynomials in $\mathbf{C}[t]$.

**Abhyankar's Question.** Let $\Omega$ be a planar semigroup. Is there a polynomial curve with $\delta$-sequence generating $\Omega$?

This question is still open. Moh [10] showed that there is no polynomial curve with $\delta$-sequence $\{6, 8, 3\}$. But there is a polynomial curve $(x, y) = (t^3, t^8)$ with $\delta$-sequence $\{3, 8\}$ which generates the same semigroup as above. Sathaye-Stenerson [16] proved that the semigroup generated by $\{6, 22, 17\}$ has no other $\delta$-sequence generating the same semigroup, and proposed the following conjecture for this question.

**Sathaye-Stenerson's Conjecture.** There is no polynomial curve having the $\delta$-sequence $\{6, 22, 17\}$.

By Algorithm 1, the defining polynomial of the curve with one place at infinity of the $\delta$-sequence $\{6, 22, 17\}$ as follows:

$$\begin{aligned} f &= (g_2^2 + a_{2,1}x^2 g_1) + c_{5,0,0}x^5 + c_{4,0,0}x^4 + c_{3,0,0}x^3 + c_{2,0,0}x^2 \\ &\quad + c_{1,1,0}xg_1 + c_{1,0,0}x + c_{0,1,0}g_1 + c_{0,0,0} \end{aligned}$$

where

$$\begin{aligned} g_1 &= y + c_3x^3 + c_2x^2 + c_1x + c_0, \\ g_2 &= (g_1^3 + a_{11}x^{11}) + c_{10,0}x^{10} + c_{9,0}x^9 + c_{8,0}x^8 + (c_{7,1}g_1 + c_{7,0})x^7 \\ &\quad + (c_{6,1}g_1 + c_{6,0})x^6 + (c_{5,1}g_1 + c_{5,0})x^5 + (c_{4,1}g_1 + c_{4,0})x^4 \\ &\quad + (c_{3,1}g_1 + c_{3,0})x^3 + (c_{2,1}g_1 + c_{2,0})x^2 + (c_{1,1}g_1 + c_{1,0})x + c_{0,1}g_1 + c_{0,0}. \end{aligned}$$

This result gives us a new approach to investigate the curve with one place at infinity of the $\delta$-sequence $\{6, 22, 17\}$ using a computer algebra system.

**6.2. Computation of moduli space.** Suzuki gave an algorithm generating the list of $\delta$-sequences of curves with one place at infinity, and implemented on a computer. From the list of $\delta$-sequences obtained by Suzuki, we could get normal forms and moduli spaces of curves with one place at infinity of genus $\leq 100$ by using the algorithm introduced in previous section. As a result, we could verify the result of Nakazawa-Oka [11].

The following is the list of moduli spaces of curves with one place at infinity for the cases genus $\leq 30$.

EXAMPLE 1. The case

`[7,[4,6,11],[2,15]]`

means that the moduli space of the curve with one place at infinity of genus 7 and the $\delta$-sequence $\{4, 6, 11\}$ is isomorphic to $(\mathbf{C}^*)^2 \times \mathbf{C}^{15}$.

```
[1,[2,3],[1,5]],
[2,[2,5],[1,8]],
[3,[2,7],[1,11]],
[3,[3,4],[1,9]],
[3,[4,6,3],[2,7]],
[4,[2,9],[1,14]],
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